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WEAKLY PERTURBED BOUNDARY-VALUE PROBLEMS FOR THE FREDHOLM INTEGRAL EQUATIONS WITH DEGENERATE KERNEL IN BANACH SPACES

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We consider weakly perturbed boundary-value problems for the Fredholm integral equations with degenerate kernel in Banach spaces and establish the conditions of bifurcation from the point $\varepsilon = 0$ for the solutions of weakly perturbed boundary-value problems for Fredholm integral equations with degenerate kernel in Banach spaces. A convergent iterative procedure is proposed for finding the solutions in the form of series $\sum_{i=-1}^{+\infty} \varepsilon^i z_i(t)$ in powers of ε.

In the present paper, we continue the investigations originated in [1] for the analysis of the conditions of solvability and the construction of solutions of weakly perturbed Fredholm integral equations with degenerate kernel in Banach spaces.

The problems of creation of efficient methods aimed at the analysis of weakly nonlinear boundary-value problems for a broad class of systems of functional-differential equations and other types of equations traditionally occupy one of the most important places in the qualitative theory of differential equations and continue the development of the methods of perturbation theory, including, in particular, the Lyapunov–Poincaré [2] and Vishik–Lyusternik [3] methods of small parameter.

The conditions of solvability of the solutions of weakly perturbed boundary-value problems for systems of ordinary differential and functional-differential equations with Noetherian linear part in Euclidean spaces and the problem of construction of their solutions were studied in [4–6].

These methods were successfully applied by Boichuk and Panasenko in [7] to the investigation of weakly perturbed boundary-value problems for systems of ordinary differential equations in Banach spaces. It is known that the differential system of linear generating ($\varepsilon = 0$) boundary-value problem possesses a solution for any right-hand side, i.e., according to the classification of S. Krein [8], it is everywhere solvable.

In [9], Boichuk and Shegda studied weakly perturbed boundary-value problems for singular differential equations that are not everywhere solvable in finite-dimensional spaces.

As a specific feature of the investigation of boundary-value problems for system of integral equations, we can mention the absence of inverse operators for the operators of their linear part [10], which significantly complicates the investigation of boundary-value problems for these equations. Hence, the problem of investigation of the conditions of appearance of the solutions of weakly perturbed boundary-value problems for the Fredholm integral equations with degenerate kernel that are not everywhere solvable in Banach spaces seems to be quite urgent.

Statement of the Problem

Let $\mathbf{C}(\mathcal{I}, \mathbf{B}_1)$ be a Banach space of vector functions $f(t)$ continuous on a finite segment $\mathcal{I} = [a, b]$ with values in the Banach space $\mathbf{B}_1$, let

$$f(t) \in \mathbf{C}(\mathcal{I}, \mathbf{B}_1) := \Big\{ f(\cdot)\colon \mathcal{I} \to \mathbf{B}_1,\ |||f||| = \sup_{t \in \mathcal{I}} \| f(t) \| \Big\},$$

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and let $\mathbf{B}$ be a Banach space.

Consider a weakly perturbed linear boundary-value problem

$$(Lz)(t) := z(t) - M(t)\int_a^b N(s)z(s)\,ds = f(t) + \varepsilon \int_a^b K(t,s)z(s)\,ds, \tag{1}$$

$$\ell z(\cdot) = \alpha + \varepsilon \ell_1 z(\cdot), \tag{2}$$

where the operator functions $M(t)$ and $N(t)$ act from the space $\mathbf{B_1}$ into $\mathbf{B_1}$ and are strongly continuous [11] with the norms

$$|||M||| = \sup_{t\in\mathcal{I}} \|M(t)\|_{\mathbf{B_1}} = M_0 < \infty \quad \text{and} \quad |||N||| = \sup_{t\in\mathcal{I}} \|N(t)\|_{\mathbf{B_1}} = N_0 < \infty,$$

the operator function $K(t,s)$ is defined in the square $\mathcal{I}\times\mathcal{I}$, acts from the Banach space $\mathbf{B_1}$ into $\mathbf{B_1}$ with respect to each variable, and is strongly continuous in t and s with the norm

$$|||K||| = \sup_{t,s\in\mathcal{I}} \|K(t,s)\|_{\mathbf{B_1}} < \infty,$$

the vector function $f(t)$ belongs to $\mathbf{C}(\mathcal{I},\mathbf{B_1})$, ℓ and ℓ_1 are linear continuous operators acting from the space $\mathbf{C}(\mathcal{I},\mathbf{B_1})$ into the Banach space $\mathbf{B}$: $\ell\colon \mathbf{C}(\mathcal{I},\mathbf{B_1}) \to \mathbf{B}$, $\ell_1\colon \mathbf{C}(\mathcal{I},\mathbf{B_1}) \to \mathbf{B}$, α is an element of the space $\mathbf{B}$: $\alpha \in \mathbf{B}$, and $\varepsilon << 1$ is a small parameter.

Assume that the generating boundary-value problem obtained from (1), (2) for $\varepsilon = 0$

$$(Lz)(t) := z(t) - M(t)\int_a^b N(s)z(s)ds = f(t), \tag{3}$$

$$\ell z(\cdot) = \alpha \tag{4}$$

has no solutions for any inhomogeneities $f(t) \in \mathbf{C}(\mathcal{I},\mathbf{B_1})$ and $\alpha \in \mathbf{B}$.

It is of interest to answer the question whether problem (1), (2) can be made solvable with the help of linear perturbations and clarify the conditions imposed on the operator function $K(t,s)$ in the integral equation (1) and the operator ℓ_1 in the boundary condition (2). To study the existence of solutions of these problems, we use the methods of the theory of generalized invertible operators.

Preliminary Information

Consider the Fredholm integral equation (3) with degenerate kernel.

Let

$$D = I_{\mathbf{B_1}} - \int_a^b N(s)M(s)\,ds$$

be a linear operator acting from the Banach space $\mathbf{B_1}$ into $\mathbf{B_1}$ bounded under the conditions imposed on the operator functions $M(t)$ and $N(t)$.

By $\mathbf{GI}(\mathbf{B}, \mathbf{B})$ we denote a class of linear bounded generalized inverse operators acting from the Banach space $\mathbf{B}$ into the Banach space $\mathbf{B}$. It is clear that the operators from $\mathbf{GI}(\mathbf{B}, \mathbf{B})$ are normally solvable [8, 12].

In [10], it is shown that if an operator D belongs to $\mathbf{GI}(\mathbf{B}_1, \mathbf{B}_1)$, then, under the condition

$$\mathcal{P}_{Y_D} \int_a^b N(s) f(s)\, ds = 0$$

and only under this condition, the operator equation (3) is solvable and has a family of solutions

$$z(t) = M(t)\mathcal{P}_{N(D)}c + (L^- f)(t), \tag{5}$$

where c is an arbitrary element of the Banach space $\mathbf{B}_1$ and

$$(L^- f)(t) = f(t) + M(t) D^- \int_a^b N(s) f(s)\, ds$$

is the bounded generalized inverse operator for the integral operator L. Here, $\mathcal{P}_{N(D)}\colon \mathbf{B}_1 \to N(D)$ is a bounded projector that projects the Banach space $\mathbf{B}_1$ onto the null space $N(D)$ of the operator D, $\mathcal{P}_{Y_D}\colon \mathbf{B}_1 \to Y_D$ is the bounded projector that projects the Banach space $\mathbf{B}_1$ onto the subspace Y_D isomorphic to the null space $N(D^*)$ of the adjoint operator D^*, and D^- is the generalized inverse operator that can be constructed by using the method proposed in [5, 6, 14].

Further, in the Banach space $\mathbf{B}_1$, we consider the linear boundary-value problem (3), (4) for the Fredholm integral equation with degenerate kernel.

We seek the solution of the boundary-value problem (3), (4) for the integral equation in the Banach space $\mathbf{C}(\mathcal{I}, \mathbf{B}_1)$ of functions $z(t)$ continuous on the segment $\mathcal{I}$ and taking values from the real Banach space $\mathbf{B}_1$. Substituting solution (5) of the inhomogeneous operator equation (3) in the boundary condition (4), we arrive at the operator equation

$$\ell(M(\cdot)\mathcal{P}_{N(D)})c + \ell f(\cdot) + \ell M(\cdot) D^- \int_a^b N(s) f(s)\, ds = \alpha. \tag{6}$$

By $Q = \ell M(\cdot)\mathcal{P}_{N(D)}$ we denote an operator acting from the Banach space $\mathbf{B}_1$ into the Banach space $\mathbf{B}$. The operator Q is bounded because it is a superposition of the bounded operator ℓ and the bounded operator function $M(t)\mathcal{P}_{N(D)}$. Thus, Eq. (6) takes the form

$$Qc = \alpha - \ell f(\cdot) - \ell M(\cdot) D^- \int_a^b N(s) f(s)\, ds.$$

Assume that the operator Q belongs to $\mathbf{GI}(\mathbf{B}_1, \mathbf{B})$. By $\mathcal{P}_{N(Q)}\colon \mathbf{B}_1 \to N(Q)$ we denote a bounded projector of the Banach space $\mathbf{B}_1$ onto the null space $N(Q)$ of the operator Q and by $\mathcal{P}_{Y_Q}\colon \mathbf{B} \to Y_Q$ we denote a bounded projector of the Banach space $\mathbf{B}$ onto the subspace $Y_Q \subset \mathbf{B}$.

Theorem 1 [13]. *Suppose that $D \in \mathbf{GI}(\mathbf{B}_1, \mathbf{B}_1)$ and $Q \in \mathbf{GI}(\mathbf{B}_1, \mathbf{B})$.*

Then the homogeneous boundary-value problem for (3), (4) with $f(t) = 0$ and $\alpha = 0$ has a family of solutions

$$z(t) = \widetilde{M}(t)c, \tag{7}$$

where

$$\widetilde{M}(t) = M(t)\mathcal{P}_{N(D)}\mathcal{P}_{N(Q)}$$

and c is an arbitrary element of the Banach space $\mathbf{B}_1$.

The inhomogeneous boundary-value problem (3), (4) is solvable for those and only those $f(t) \in \mathbf{C}(\mathcal{I}, \mathbf{B}_1)$ and $\alpha \in \mathbf{B}$ that satisfy the system of conditions

$$\mathcal{P}_{Y_D} \int\limits_a^b N(s) f(s)\, ds = 0,$$

$$\mathcal{P}_{Y_Q} \left[\alpha - \ell f(\cdot) - \ell M(\cdot) D^- \int\limits_a^b N(s) f(s) ds \right] = 0,$$

and, in addition, possesses a family of solutions

$$z(t) = \widetilde{M}(t)c + (Gf)(t) + M(t)\mathcal{P}_{N(D)} Q^- \alpha,$$

where

$$(Gf)(t) := \left[f(t) - M(t)\mathcal{P}_{N(D)} Q^- \ell f(\cdot) \right]$$

$$+ M(t) \left[I_{\mathbf{B}_1} - \mathcal{P}_{N(D)} Q^- \ell M(\cdot) \right] D^- \int\limits_a^b N(s) f(s)\, ds \tag{8}$$

is the generalized Green operator for the semihomogeneous boundary-value problem (3), (4) with $\alpha = 0$.

Main Result

To solve the posed problem, we use the Vishik–Lyusternik method [3] and establish the conditions of appearance of solutions of the boundary-value problem (1), (2) in the form of a part of the series

$$z(t, \varepsilon) = \sum_{i=-1}^{+\infty} \varepsilon^i z_i(t) \tag{9}$$

in powers of the small parameter ε containing a negative power of ε.

We substitute series (9) in the boundary-value problem (3), (4) and equate the coefficients of the same powers of ε.

For ε^{-1}, we arrive at the following homogeneous boundary-value problem:

$$z_{-1}(t) - M(t)\int_a^b N(s)z_{-1}(s)\,ds = 0, \tag{10}$$

$$\ell z_{-1}(\cdot) = 0 \tag{11}$$

for the determination of $z_{-1}(t)$.

By Theorem 1, the homogeneous boundary-value problem (10), (11) possesses a solution

$$z_{-1}(t, c_{-1}) = \widetilde{M}(t)c_{-1}, \tag{12}$$

where $c_{-1} \in \mathbf{B_1}$ is an arbitrary element determined in what follows.

Equating the coefficients of ε^0, we arrive at the boundary-value problem

$$z_0(t) - M(t)\int_a^b N(s)z_0(s)\,ds = f(t) + \int_a^b K(t,s)z_{-1}(s)\,ds, \tag{13}$$

$$\ell z_0(\cdot) = \alpha + \ell_1 z_{-1}(\cdot) \tag{14}$$

for the coefficient $z_0(t)$.

By Theorem 1, the linear inhomogeneous boundary-value problem (13), (14) is solvable if and only if the system of conditions

$$\mathcal{P}_{Y_D}\int_a^b N(s)\left[f(s) + \int_a^b K(s,\tau)z_{-1}(\tau)\,d\tau\right]ds = 0,$$

$$\mathcal{P}_{Y_Q}\left\{\alpha + \ell_1\widetilde{M}(\cdot)c_{-1} - \ell\left[f(\cdot) + \int_a^b K(\cdot,s)z_{-1}(s)ds\right]\right.$$

$$\left. -\ell M(\cdot)D^{-}\int_a^b N(s)\left[f(s) + \int_a^b K(s,\tau)z_{-1}(\tau)\,d\tau\right]ds\right\} = 0$$

is satisfied. Substituting $z_{-1}(t, c_{-1})$ from (12), we obtain the system of equations

$$\mathcal{P}_{Y_D}\int_a^b N(s)\left[f(s) + \int_a^b K(s,\tau)\widetilde{M}(\tau)c_{-1}d\tau\right]ds = 0, \tag{15}$$

$$\mathcal{P}_{Y_Q}\left\{\alpha + \ell_1\widetilde{M}(\cdot)c_{-1} - \ell\left[f(\cdot) + \int_a^b K(\cdot,s)\widetilde{M}(s)c_{-1}\,ds\right]\right.$$

$$-\ell M(\cdot)D^{-}\int\limits_a^b N(s)\left[f(s)+\int\limits_a^b K(s,\tau)\widetilde{M}(\tau)c_{-1}d\tau\right]ds\Biggr\} = 0.$$

Denoting

$$B_0 = \begin{bmatrix} \mathcal{P}_{Y_D}\displaystyle\int_a^b N(s)\int_a^b K(s,\tau)\widetilde{M}(\tau)\,d\tau\,ds \\ \mathcal{P}_{Y_Q}\Biggl\{\ell_1\widetilde{M}(\cdot)-\ell\displaystyle\int_a^b K(\cdot,s)\widetilde{M}(s)\,ds \\ -\ell M(\cdot)D^{-}\displaystyle\int_a^b N(s)\int_a^b K(s,\tau)\widetilde{M}(\tau)\,d\tau\,ds\Biggr\} \end{bmatrix}, \tag{16}$$

we derive the following operator equation for the element $c_{-1} \in \mathbf{B_1}$ from (15):

$$B_0c_{-1} = -\begin{bmatrix} \mathcal{P}_{Y_D}\displaystyle\int_a^b N(s)f(s)\,ds \\ \mathcal{P}_{Y_Q}\Biggl\{\alpha-\ell f(\cdot)-\ell M(\cdot)D^{-}\displaystyle\int_a^b N(s)f(s)\,ds\Biggr\} \end{bmatrix}. \tag{17}$$

The operator B_0 acts from the Banach space $\mathbf{B_1}$ into the direct product of Banach spaces $\mathbf{B_1}\times\mathbf{B}$. Assume that the operator B_0 belongs to $\mathbf{GI(B_1, B_1\times B)}$. Then it is normally solvable and there exist bounded projectors

$$\mathcal{P}_{N(B_0)}\colon \mathbf{B_1}\to N(B_0) \quad \text{and} \quad \mathcal{P}_{Y_{B_0}}\colon \mathbf{B_1}\times\mathbf{B}\to Y_{B_0}$$

and the bounded generalized inverse operator $B_0^{-}\colon \mathbf{B_1}\times\mathbf{B}\to\mathbf{B_1}$ for the operator B_0.

In view of the normal solvability of the operator B_0, Eq. (17) has a solution if and only if its right-hand side satisfies the conditions

$$\mathcal{P}_{Y_{B_0}}\begin{bmatrix} \mathcal{P}_{Y_D}\displaystyle\int_a^b N(s)f(s)\,ds \\ \mathcal{P}_{Y_Q}\Biggl\{\alpha-\ell f(\cdot)-\ell M(\cdot)D^{-}\displaystyle\int_a^b N(s)f(s)\,ds\Biggr\} \end{bmatrix} = 0.$$

The last condition is satisfied provided that the condition

$$\mathcal{P}_{Y_{B_0}}\begin{bmatrix} \mathcal{P}_{Y_D} \\ \mathcal{P}_{Y_Q} \end{bmatrix} = 0 \tag{18}$$

is satisfied and, in addition, the operator equation (17) possesses at least one solution

$$c_{-1} = -B_0^- \begin{bmatrix} \mathcal{P}_{Y_D} \int_a^b N(s) f(s)\, ds \\ \mathcal{P}_{Y_Q} \left\{ \alpha - \ell f(\cdot) - \ell M(\cdot) D^- \int_a^b N(s) f(s)\, ds \right\} \end{bmatrix}.$$

Substituting the obtained c_{-1} in (12), we get the solution

$$z_{-1}(t, c_{-1}) = -\widetilde{M}(t) B_0^- \begin{bmatrix} \mathcal{P}_{Y_D} \int_a^b N(s) f(s)\, ds \\ \mathcal{P}_{Y_Q} \left\{ \alpha - \ell f(\cdot) - \ell M(\cdot) D^- \int_a^b N(s) f(s)\, ds \right\} \end{bmatrix}$$

of the boundary-value problem (10), (11). Moreover, the boundary-value problem (13), (14) possesses a family of solutions

$$z_0(t, c_0) = \widetilde{M}(t) c_0 + \bar{z}_0(t), \tag{19}$$

where $c_0 \in \mathbf{B_1}$ is an arbitrary element of the space $\mathbf{B_1}$ determined in the next step of the iterative process,

$$\bar{z}_0(t) = M(t) \mathcal{P}_{N(D)} Q^- \left[\alpha + \ell_1 z_{-1}(\cdot, c_{-1})\right] + \left(G \left[f(\cdot) + \int\limits_a^b K(\cdot, s) z_{-1}(s)\, ds \right] \right)(t),$$

and G is the generalized Green operator (8).

The generalized Green operator G of the boundary-value problem (13), (14) acts upon the operator function

$$f(t) + \int\limits_a^b K(t, s) z_{-1}(s)\, ds$$

by the rule

$$\begin{aligned} \left(G \left[f(\cdot) + \int\limits_a^b K(\cdot, s) z_{-1}(s) ds \right] \right)(t) := f(t) + \int\limits_a^b K(t, s) z_{-1}(s)\, ds \\ - M(t) \mathcal{P}_{N(D)} Q^- \ell \left[f(\cdot) + \int\limits_a^b K(\cdot, s) z_{-1}(s)\, ds \right] \\ + M(t) \left[I_{\mathbf{B}_1} - \mathcal{P}_{N(D)} Q^- \ell M(\cdot) \right] \end{aligned}$$

$$\times D^{-}\int_a^b N(s)\left[f(s)+\int_a^b K(s,\tau)z_{-1}(s)\,d\tau\right]ds.$$

For ε^1, we arrive at the following boundary-value problem for the coefficient $z_1(t)$:

$$z_1(t)-M(t)\int_a^b N(s)z_1(s)\,ds=\int_a^b K(t,s)z_0(s)\,ds, \tag{20}$$

$$\ell z_1(\cdot)=\ell_1 z_0(\cdot). \tag{21}$$

By Theorem 1, the linear inhomogeneous boundary-value problem (20), (21) is solvable if and only if the system of conditions

$$\mathcal{P}_{Y_D}\int_a^b N(s)\int_a^b K(s,\tau)z_0(\tau)\,d\tau\,ds=0,$$

$$\mathcal{P}_{Y_Q}\left\{\ell_1\left[\widetilde{M}(\cdot)c_0+\bar{z}_0(\cdot)\right]-\ell\int_a^b K(\cdot,s)z_0(s)ds\right.$$
$$\left.-\ell M(\cdot)D^{-}\int_a^b N(s)\int_a^b K(s,\tau)z_0(\tau)\,d\tau\,ds\right\}=0$$

is satisfied. Substituting $z_0(t,c_0)$ from (19) and using (16), we get the following operator equation for the element $c_0\in\mathbf{B_1}$:

$$B_0c_0=-\begin{bmatrix}\mathcal{P}_{Y_D}\int_a^b N(s)\int_a^b K(s,\tau)\bar{z}_0(\tau)\,d\tau\,ds\\ \mathcal{P}_{Y_Q}\left\{\ell_1\bar{z}_0(\cdot)-\ell M(\cdot)D^{-}\int_a^b N(s)\int_a^b K(s,\tau)\bar{z}_0(\tau)\,d\tau\,ds\right\}\end{bmatrix}. \tag{22}$$

In view of the normal solvability of the operator B_0, Eq. (22) possesses a solution if and only if its right-hand side satisfies the condition

$$\mathcal{P}_{Y_{B_0}}\begin{bmatrix}\mathcal{P}_{Y_D}\int_a^b N(s)\int_a^b K(s,\tau)\bar{z}_0(\tau)\,d\tau\,ds\\ \mathcal{P}_{Y_Q}\left\{\ell_1\bar{z}_0(\cdot)-\ell M(\cdot)D^{-}\int_a^b N(s)\int_a^b K(s,\tau)\bar{z}_0(\tau)\,d\tau\,ds\right\}\end{bmatrix}=0.$$

In the case where (18) is true, this condition is satisfied and the operator equation (22) has at least one solution

$$c_0 = -B_0^- \begin{bmatrix} \mathcal{P}_{Y_D} \displaystyle\int_a^b N(s) \int_a^b K(s,\tau)\bar{z}_0(\tau)\,d\tau\,ds \\ \mathcal{P}_{Y_Q}\left\{\ell_1 \bar{z}_0(\cdot) - \ell M(\cdot) D^- \displaystyle\int_a^b N(s)\int_a^b K(s,\tau)\bar{z}_0(\tau)\,d\tau\,ds\right\} \end{bmatrix}.$$

Substituting c_0 in (19), we obtain the following solution of the boundary-value problem (13), (14):

$$z_0(t,c_0) = -\widetilde{M}(t) B_0^- \begin{bmatrix} \mathcal{P}_{Y_D} \displaystyle\int_a^b N(s) \int_a^b K(s,\tau)\bar{z}_0(\tau)\,d\tau\,ds \\ \mathcal{P}_{Y_Q}\left\{\ell_1 \bar{z}_0(\cdot) - \ell M(\cdot) D^- \displaystyle\int_a^b N(s)\int_a^b K(s,\tau)\bar{z}_0(\tau)\,d\tau\,ds\right\} \end{bmatrix} + \bar{z}_0(t).$$

Moreover, the boundary-value problem (20), (21) has a family of solutions

$$z_1(t,c_1) = \widetilde{M}(t)c_1 + \bar{z}_1(t), \tag{23}$$

where

$$\bar{z}_1(t) = M(t)\mathcal{P}_{N(D)}Q^-\ell_1 z_0(\cdot) + \left(G\left[f(\cdot) + \int\limits_a^b K(\cdot,s)z_0(s)ds\right]\right)(t),$$

G is the generalized Green operator (8), and $c_1 \in \mathbf{B_1}$ is an arbitrary element of the space $\mathbf{B_1}$ determined in the next step of the iterative process.

For ε^2, we get the following boundary-value problem for the coefficient $z_2(t)$:

$$z_2(t) - M(t)\int\limits_a^b N(s)z_2(s)\,ds = \int\limits_a^b K(t,s)z_1(s)\,ds, \tag{24}$$

$$\ell z_2(\cdot) = \ell_1 z_1(\cdot). \tag{25}$$

By Theorem 1, the inhomogeneous linear boundary-value problem (24), (25) is solvable if and only if the system of conditions

$$\mathcal{P}_{Y_D}\int\limits_a^b N(s)\int\limits_a^b K(s,\tau)z_1(\tau)d\tau\,ds = 0,$$

$$\mathcal{P}_{Y_Q}\left\{\ell_1\left[\widetilde{M}(\cdot)c_1 + \bar{z}_1(\cdot)\right] - \ell\int\limits_a^b K(\cdot,s)z_1(s)\,ds\right.$$

$$-\ell M(\cdot)D^{-}\int\limits_a^b N(s)\int\limits_a^b K(s,\tau)z_1(\tau)\,d\tau\,ds\Bigg\} = 0$$

is satisfied. Substituting $z_1(t,c_1)$ from (23) and using (16), we arrive at the following operator equation for the element $c_1 \in \mathbf{B_1}$:

$$B_0c_1 = -\begin{bmatrix} \mathcal{P}_{Y_D}\displaystyle\int_a^b N(s)\int_a^b K(s,\tau)\bar{z}_1(\tau)\,d\tau\,ds \\ \mathcal{P}_{Y_Q}\left\{\ell_1\bar{z}_1(\cdot) - \ell M(\cdot)D^{-}\displaystyle\int_a^b N(s)\int_a^b K(s,\tau)\bar{z}_1(\tau)\,d\tau\,ds\right\} \end{bmatrix}. \tag{26}$$

In view of the normal solvability of the operator B_0, Eq. (26) is solvable if and only if its right-hand side satisfies the conditions

$$\mathcal{P}_{Y_{B_0}}\begin{bmatrix} \mathcal{P}_{Y_D}\displaystyle\int_a^b N(s)\int_a^b K(s,\tau)\bar{z}_1(\tau)\,d\tau\,ds \\ \mathcal{P}_{Y_Q}\left\{\ell_1\bar{z}_1(\cdot) - \ell M(\cdot)D^{-}\displaystyle\int_a^b N(s)\int_a^b K(s,\tau)\bar{z}_1(\tau)\,d\tau\,ds\right\} \end{bmatrix} = 0.$$

Under condition (18), these conditions are satisfied and the operator equation (26) has at least one solution

$$c_1 = -B_0^{-}\begin{bmatrix} \mathcal{P}_{Y_D}\displaystyle\int_a^b N(s)\int_a^b K(s,\tau)\bar{z}_1(\tau)\,d\tau\,ds \\ \mathcal{P}_{Y_Q}\left\{\ell_1\bar{z}_1(\cdot) - \ell M(\cdot)D^{-}\displaystyle\int_a^b N(s)\int_a^b K(s,\tau)\bar{z}_1(\tau)\,d\tau\,ds\right\} \end{bmatrix}.$$

Substituting the obtained c_1 in (23), we get the following solution of the boundary-value problem (20), (21):

$$z_1(t) = -\widetilde{M}(t)B_0^{-}\begin{bmatrix} \mathcal{P}_{Y_D}\displaystyle\int_a^b N(s)\int_a^b K(s,\tau)\bar{z}_1(\tau)\,d\tau\,ds \\ \mathcal{P}_{Y_Q}\left\{\ell_1\bar{z}_1(\cdot) - \ell M(\cdot)D^{-}\displaystyle\int_a^b N(s)\int_a^b K(s,\tau)\bar{z}_1(\tau)\,d\tau\,ds\right\} \end{bmatrix} + \bar{z}_1(t).$$

Moreover, the boundary-value problem (24), (25) possesses a family of solutions

$$z_2(t,c_2) = \widetilde{M}(t)c_2 + \bar{z}_2(t),$$

where

$$\bar{z}_2(t) = M(t)\mathcal{P}_{N(D)}Q^{-}\ell_1 z_1(\cdot) + \left(G\left[f(\cdot) + \int\limits_a^b K(\cdot, s)z_1(s)\,ds\right]\right)(t),$$

G is the generalized Green operator (8), and c_2 is an arbitrary element of the space $\mathbf{B_1}$ determined in the next step of the iterative process.

By induction, we obtain the following boundary-value problems for the coefficients $z_i(t)$ of ε^i of series (9):

$$z_i(t) - M(t)\int\limits_a^b N(s)z_i(s)\,ds = \int\limits_a^b K(t,s)z_{i-1}(s)\,ds, \tag{27}$$

$$\ell z_i(\cdot) = \ell_1 z_{i-1}(\cdot). \tag{28}$$

By Theorem 1, the linear inhomogeneous boundary-value problems (27), (28) are solvable if and only if the system of conditions

$$\mathcal{P}_{Y_D}\int\limits_a^b N(s)\int\limits_a^b K(s,\tau)z_{i-1}(\tau)\,d\tau\,ds = 0,$$

$$\mathcal{P}_{Y_Q}\left\{\ell_1\left[\widetilde{M}(\cdot)c_{i-1} + \bar{z}_{i-1}(\cdot)\right] - \ell\int\limits_a^b K(\cdot,s)z_{i-1}(s)ds\right.$$

$$\left. -\ell M(\cdot)D^{-}\int\limits_a^b N(s)\int\limits_a^b K(s,\tau)z_{i-1}(\tau)\,d\tau\,ds\right\} = 0$$

is satisfied. Substituting

$$z_{i-1}(t, c_{i-1}) = \widetilde{M}(t)c_{i-1} + \bar{z}_{i-1}(t) \tag{29}$$

and using (16), we arrive at the operator equations for the elements $c_{i-1} \in \mathbf{B_1}$:

$$B_0 c_{i-1} = -\begin{bmatrix} \mathcal{P}_{Y_D}\int_a^b N(s)\int_a^b K(s,\tau)\bar{z}_{i-1}(\tau)\,d\tau\,ds \\ \\ \mathcal{P}_{Y_Q}\left\{\ell_1\bar{z}_{i-1}(\cdot) - \ell M(\cdot)D^{-}\int_a^b N(s)\int_a^b K(s,\tau)\bar{z}_{i-1}(\tau)\,d\tau\,ds\right\} \end{bmatrix}. \tag{30}$$

In view of the normal solvability of the operator B_0, Eqs. (30) are solvable if and only if

$$\mathcal{P}_{Y_{B_0}} \begin{bmatrix} \mathcal{P}_{Y_D} \int_a^b N(s) \int_a^b K(s,\tau)\bar{z}_{i-1}(\tau)\, d\tau\, ds \\ \\ \mathcal{P}_{Y_Q} \left\{ \ell_1 \bar{z}_{i-1}(\cdot) - \ell M(\cdot) D^- \int_a^b N(s) \int_a^b K(s,\tau)\bar{z}_{i-1}(\tau)\, d\tau\, ds \right\} \end{bmatrix} = 0. \tag{31}$$

Under condition (18), conditions (31) are satisfied and each of the operator equations (30) possesses at least one solution

$$c_{i-1} = -B_0^- \begin{bmatrix} \mathcal{P}_{Y_D} \int_a^b N(s) \int_a^b K(s,\tau)\bar{z}_{i-1}(\tau)\, d\tau\, ds \\ \\ \mathcal{P}_{Y_Q} \left\{ \ell_1 \bar{z}_0(\cdot) - \ell M(\cdot) D^- \int_a^b N(s) \int_a^b K(s,\tau)\bar{z}_{i-1}(\tau)\, d\tau\, ds \right\} \end{bmatrix}.$$

Substituting c_{i-1} in (29), we obtain solutions of the boundary-value problems (27), (28):

$$z_{i-1}(t) = -\widetilde{M}(t) B_0^-$$

$$\times \begin{bmatrix} \mathcal{P}_{Y_D} \int_a^b N(s) \int_a^b K(s,\tau)\bar{z}_{i-1}(\tau)\, d\tau\, ds \\ \\ \mathcal{P}_{Y_Q} \left\{ \ell_1 \bar{z}_{i-1}(\cdot) - \ell M(\cdot) D^- \int_a^b N(s) \int_a^b K(s,\tau)\bar{z}_{i-1}(\tau)\, d\tau\, ds \right\} \end{bmatrix} + \bar{z}_{i-1}(t),$$

where

$$\bar{z}_i(t) = M(t)\mathcal{P}_{N(D)} Q^- \ell_1 z_{i-1}(\cdot) + \left(G \int_a^b K(\cdot, s) z_{i-1}(s)\, ds \right)(t).$$

Thus, we get the following iterative algorithm for the construction of solution of the boundary-value problem (1), (2):

$$z_i(t, c_i) = \begin{bmatrix} M(t)\mathcal{P}_{N(D)}\mathcal{P}_{N(Q)} c_{-1} & \text{for} \quad i = -1, \\ \\ M(t)\mathcal{P}_{N(D)}\mathcal{P}_{N(Q)} c_i + \bar{z}_i(t) & \text{for} \quad i = \overline{0, \infty}, \end{bmatrix} \tag{32}$$

$$c_i = \begin{cases} -B_0^- \begin{bmatrix} \mathcal{P}_{Y_D} \displaystyle\int_a^b N(s) f(s) ds \\ \mathcal{P}_{Y_Q} \Big\{ \alpha - \ell f(\cdot) - \ell M(\cdot) D^- \\ \times \displaystyle\int_a^b N(s) f(s)\, ds \Big\} \end{bmatrix} & \text{for} \quad i = -1, \\ -B_0^- \begin{bmatrix} \mathcal{P}_{Y_D} \displaystyle\int_a^b N(s) \int_a^b K(s,\tau) \bar{z}_i(\tau)\, d\tau\, ds \\ \mathcal{P}_{Y_Q} \Big\{ \ell_1 \bar{z}_i(\cdot) - \ell M(\cdot) D^- \displaystyle\int_a^b N(s) \\ \times \displaystyle\int_a^b K(s,\tau) \bar{z}_i(\tau) d\tau ds \Big\} \end{bmatrix} & \text{for} \quad i = \overline{0, \infty}, \end{cases} \tag{33}$$

$$\bar{z}_i(t) = \begin{cases} M(t) \mathcal{P}_{N(D)} Q^- [\alpha + \ell_1 z_{-1}(\cdot, c_{-1})] \\ \quad + \left(G \left[f(\cdot) + \displaystyle\int_a^b K(\cdot, s) z_{-1}(s)\, ds \right] \right)(t) & \text{for} \quad i = 0, \\ M(t) \mathcal{P}_{N(D)} Q^- \ell_1 z_{i-1}(\cdot) \\ \quad + \left(G \displaystyle\int_a^b K(\cdot, s) z_{i-1}(s)\, ds \right)(t) & \text{for} \quad i = \overline{1, \infty}, \end{cases} \tag{34}$$

$$\left(G \int_a^b K(\cdot, s) z_{i-1}(s)\, ds \right)(t)$$

$$:= \begin{cases} f(t) + \displaystyle\int_a^b K(t,s) z_{-1}(s)\, ds - M(t) \mathcal{P}_{N(D)} Q^- \ell \\ \quad \times \left[f(\cdot) + \displaystyle\int_a^b K(\cdot, s) z_{-1}(s)\, ds \right] \\ \quad + M(t) \left[I_{\mathbf{B}_1} - \mathcal{P}_{N(D)} Q^- \ell M(\cdot) \right] \\ \quad \times D^- \displaystyle\int_a^b N(s) \left[f(s) + \int_a^b K(s,\tau) z_{-1}(s)\, d\tau \right] ds & \text{for} \quad i = 0, \\ \displaystyle\int_a^b K(t,s) z_{i-1}(s)\, ds - M(t) \mathcal{P}_{N(D)} Q^- \\ \quad \times \ell \displaystyle\int_a^b K(\cdot, s) z_{i-1}(s)\, ds + M(t) \left[I_{\mathbf{B}_1} - \mathcal{P}_{N(D)} Q^- \ell M(\cdot) \right] \\ \quad \times D^- \displaystyle\int_a^b N(s) \int_a^b K(s,\tau) z_{-1}(s)\, d\tau\, ds & \text{for} \quad i = \overline{1, \infty}. \end{cases} \tag{35}$$

The convergence of series (9) can be proved by using the method proposed in [1, 5, 7].

Theorem 2. *Suppose that* $D \in \mathbf{GI}(\mathbf{B_1}, \mathbf{B_1})$, $Q \in \mathbf{GI}(\mathbf{B_1}, \mathbf{B})$, *and the generating boundary-value problem (3), (4) does not have solutions for arbitrary inhomogeneities* $f(t) \in \mathbf{C}(\mathcal{I}, \mathbf{B_1})$ *and* $\alpha \in \mathbf{B}$.

If the operator

$$B_0 \in \mathbf{GI}(\mathbf{B_1}, \mathbf{B_1} \times \mathbf{B})$$

and conditions (18) are satisfied, then, for any inhomogeneities $f(t) \in \mathbf{C}(\mathcal{I}, \mathbf{B_1})$ *and* $\alpha \in \mathbf{B}$, *the weakly perturbed boundary-value problem (1), (2) possesses a family of solutions in the form of a series*

$$z(t, \varepsilon) = \sum_{i=-1}^{+\infty} \varepsilon^i z_i(t)$$

absolutely convergent for any fixed $\varepsilon \in (0, \varepsilon_*]$ *whose coefficients are determined by using the iterative algorithm (32)–(35).*

Remark 1. If $\mathcal{P}_{N(B_0)} = 0$, then the operator equations (17), (22), etc., are n-normal and uniquely solvable in each step of the iterative process [14]. Moreover, the operator B_0^- is the left inverse operator $(B_0)_l^{-1}$.

Thus, in the case where conditions (18) are satisfied, the boundary-value problem (1), (2) possesses a family of solutions in the form of series (4) whose coefficients are determined by the iterative algorithm (32)–(35) in which

$$B_0^- = (B_0)_l^{-1}.$$

Remark 2. If $\mathcal{P}_{Y_{B_0}} = 0$, then the operator equations (17), (22), etc., are d-normal and everywhere solvable in each step of the iterative process [14]. Moreover, the operator B_0^- is the right inverse operator $(B_0)_r^{-1}$.

Then conditions (18) are always satisfied and, for any inhomogeneities $f(t) \in \mathbf{C}(\mathcal{I}, \mathbf{B_1})$, the boundary-value problem (1), (2) possesses a family of solutions in the form of series (4) whose coefficients are determined by the iterative algorithm (32)–(35) with

$$B_0^- = (B_0)_r^{-1}.$$

Remark 3. Conditions (18) are sufficient for the existence of solution of the boundary-value problem (1), (2). If these conditions are not satisfied, then the solution of the boundary-value problem (1), (2) in the form of series (4) does not exist. However, the solution of the boundary-value problem (1), (2) may exist in the form of a series

$$\sum_{i=-2}^{+\infty} \varepsilon^i z_i(t).$$

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