On the classification of the serial principal posets

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A finite poset S is called principal if the quadratic Tits form $q_S(z) := z_0^2 + \sum_{i\in S} z_i^2 + \sum_{i<j,i,j\in S} z_i z_j - z_0 \sum_{i\in S} z_i$ of S is non-negative and $\operatorname{Ker} q_S(z) := \{t \mid q_S(t) = 0\}$ is an infinite cyclic group, i.e. $\operatorname{Ker} q_S(z) = t_0\mathbb{Z}$ for some $t_0 \neq 0$. We call a principal poset S serial if for any $m \in \mathbb{N}$, there is a principal poset $S(m) \supset S$ such that $|S(m) \setminus S| = m$.

By a subposet we always mean a full subposet. A poset S is called a sum of subposets A and B if $S = A \cup B$ and $A \cap B = \varnothing$. If any two elements $a \in A$ and $b \in B$ are incomparable, the sum is called direct. A sum $S = A + B$ with $A, B \neq \varnothing$ is said to be left (resp. right) if $a < b$ (resp. $b < a$) for some $a \in A, b \in B$ and there is no $a' \in A, b' \in B$ satisfying $a' > b'$ (resp. $b' > a'$). Both left and right sums are called one-sided. A sum $S = A + B$ is called two-sided if $a < b$ and $a' > b'$ for some $a, a' \in A, b, b' \in B$. Finally, a one-sided or two-sided sum $S = A + B$ is called minimax if $x < y$ with x and y belonging to different summands implies that x is minimal and y maximal in S.

We can now formulate our main theorems.

Theorem 1. *A poset S is serial principal if and only if one of the following condition holds*:

(I) *S is a direct sum of a chain of length $k \geq 0$, and a semichain of length $s \geq 2$ and 2-length 2;*

(II) *S is a direct sum of a semichain of length $k \geq 1$ and 2-length 1, and a semichain of length $s \geq 1$ and 2-length 1, where $k \leq s$;*

(III) *S is a left minimax sum of a chain of length $k \geq 1$, and a semichain of length $s \geq 2$ and 2-length 1 with the only maximal element;*

(IV) *S is a left minimax sum of a semichain of length $k \geq 2$ and 2-length 1 with the only minimal element, and a chain of length $s \geq 1$;*

(V) *S is a two-sided minimax sum of a chain of length $k \geq 2$ and a chain of length $s \geq 3$, where $k \leq s$.*

Moreover, all these posets are pairwise non-isomorphic.

Theorem 2. *Any principal poset of order $n > 8$ is serial.*

A class of principal posets of order $n = 6, 7, 8$ (which in our terminology means the non-serial ones) were written by G. Marczak, D. Simson and K. Zając with the help of programming in Maple and Python in the paper [**1**] and the preprint [**2**].

References

1. G. Marczak, D. Simson and K. Zajac, *Algorithmic computation of principal posets using Maple and Python*, Algebra and Discr. Math. **17** (2014), 33–69.
2. G. Marczak, D. Simson and K. Zajac, *Tables of one-peak principal posets of Coxeter-Euclidean type $\tilde{E}_8$*, URL: //http: www.umk. pl/ mgasiorek/pdf/OnePeakPrincipal PosetsE8Tables.pdf.

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