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RESEARCH ARTICLE

On finite posets of *inj*-finite type and their Tits forms

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Abstract. We prove one theorem on connection between *inj*-finiteness of a finite poset and positive definiteness of its Tits form.

The quadratic Tits form, introduced by P. Gabriel [1] for quivers, S. Brenner [2] for quivers with relations, Yu. A. Drozd [3] for posets, etc. plays an important role in representation theory (see e.g. the introduction to [4]). In particular, in [2] it is proved that a finite poset A (without an element 0) has only finitely many isomorphism classes of indecomposable representations (over a field) if and only if its Tits form $q_A : \mathbb{Z}^{A\cup 0} \to \mathbb{Z}$, defined by the equality

$$q_A(z) = z_0^2 + \sum_{i\in A} z_i^2 + \sum_{\substack{i<j,\\ i,j\in A}} z_i z_j - z_0 \sum_{i\in A} z_i,$$

is weakly positive (a quadratic form $f(z) : \mathbb{Z}^n \to \mathbb{Z}$ is called weakly positive if $f(z) > 0$ for all nonzero $z = (z_1, \dots, z_n)$ with $z_1, \dots, z_n \geq 0$; if $f(z) > 0$ for all $z \neq 0$, the form is called positive definite or simply positive).

In [5] the authors proved that the category of representations of the category $InjA$ (of injective representations of A) has only finitely many isomorphism classes of indecomposable objects if and only if the Tits form of $InjA$ is weakly positive. In this paper we continue to study the categories $InjA$ of such type.

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1. Injective representations of posets

Throughout the paper, all posets (partially ordered sets) are finite and all vector spaces are finite-dimensional; k denotes a fixed field.

Recall some well-known definitions in terms of vector spaces graded by posets (see [5], [6]).

Let A be a poset. An A-graded k-vector space is by definition the direct sum $U = \bigoplus_{a \in A} U_a$ of k-vector spaces U_a. A linear map $\varphi : U \to U'$ between A-graded vector spaces U and U'

is called an A-map if $\varphi_{a^*a^*} = \varphi_{aa}$ for each $a \in A$ and $\varphi_{bc} = 0$ for each $b, c \in A$ not satisfying $b \leqslant c$, where φ_{xy} denotes the linear map of U_x into U'_y induced by the map φ.

A representation of a poset A over k is by definition a triple $W = (V, U, \gamma)$ formed by a k-vector space V, an A-graded k-vector space U and a linear map $\gamma : V \to U$; a morphism of representations

$W \to W'$ is given by a pair (μ, ν), formed by a linear map $\mu : V \to V'$ and an A-map $\nu : U \to U'$ such that $\gamma\nu = \mu\gamma'$. The category of representations of A will be denoted by *RepA*. For a morphism $\alpha = (\mu, \nu) : X \to Y$ in *RepA*, we write $0 \Rightarrow X \overset{\alpha}{\to} Y$ if μ and all ν_{xx} are injective. A representation X of a poset A is said to be injective if any diagram

$$\begin{array}{ccccc} 0 & \Rightarrow & R' & \to & R \\ & & \downarrow & & \\ & & X & & \end{array}$$

can be embedded in a commutative diagram

$$\begin{array}{ccccc} 0 & \Rightarrow & R' & \to & R \\ & & \downarrow & \swarrow & \\ & & X & & \end{array} .$$

The full subcategory of *RepA* consisting of all injective objects will be denoted by $InjA$. We say that A is of *inj*-finite type if the category $Funct(InjA, \operatorname{mod} k)$ of representations of $InjA$ has only finitely many isomorphism classes of indecomposable objects.

We associate to a poset A the quiver $\overrightarrow{A} = (\overrightarrow{A}_0, \overrightarrow{A}_1)$ with the set of vertices $\overrightarrow{A}_0 = A$ and the set of arrows

$$\overrightarrow{A}_1 = \{i \to j \mid i < j,\ i \text{ and } j \text{ are adjacent}\}$$

(elements i and $j > i$ is called adjacent if there is not an element s, such that $j > s > i$). We will consider $\overrightarrow{A}$ as a commutative quiver, that is, any two non-trivial path in $\overrightarrow{A}$ with the same starting and terminating vertices are equal (and $\overrightarrow{A}$ has no other relations). An arrow $x \to y$ is denoted by (x, y), and we write $[x, y]$ if there is an arrow $x \to y$ or $y \to x$.

In [5] the authors proved the following theorem.

Theorem 1. *Let A be a poset and $B = A \cup \infty$, where $x < \infty$ for any $x \in A$. The poset A is of inj-finite type if and only if the commutative quiver $\overrightarrow{B}$ contains no subquiver (with relations) isomorphic or antiisomorphic to one of the following connected commutative quiver $Q = (Q_0, Q_1)$:*

I. $Q_1 = \{(1,3),(1,4),(2,3),(2,4)\}$;

II. $Q_1 = \{[1,2],[1,3],[1,4],[1,5]\}$;

III. $Q_1 = \{[1,2],[2,3],[1,4],[4,5],[1,6],[6,7]\}$;

IV. $Q_1 = \{[1,2],[2,3],[3,4],[1,5],[5,6],[6,7],[1,8]\}$;

V. $Q_1 = \{[1,2],[2,3],[1,4],[4,5],[5,6],[6,7],[7,8],[1,9]\}$;

VI. $Q_1 = \{[1,2],[2,3],[3,4],[4,5],(6,5),(5,8),(6,7),(7,8),[7,9]\}$;

VII. $Q_1 = \{[1,2],[2,3],[3,4],[4,5],(6,5),(5,8),(6,7),(7,8),[8,9]\}$;

VIII. $Q_1 = \{[1,2],[2,3],[3,4],[4,5],(6,5),(7,6),(8,5),(7,8),[8,9]\}$;

IX. $Q_1 = \{[1,2],[2,3],(4,3),(3,8),(4,5),(5,8),[5,6],[6,7]\}$;

X. $Q_1 = \{[1,2],[2,3],[3,4],(5,4),(4,8),(5,6),(6,7),(7,8),[7,9]\}$;

XI. $Q_1 = \{(1,2),(2,5),(1,3),(3,4),(4,5),[4,6],[6,7],[7,8],[8,9]\}$;

XII. $Q_1 = \{[1,2],[2,3],(4,3),(3,8),(4,5),(5,6),(6,7),(7,8),[6,9]\}$.

Recall that a quiver Q is called a subquiver of a commutative quiver P if it can be obtained from P by combination of the following operations:

a) rejection of a (+)- or (-)-admissible vertex (i.e., such that is not, respectively, starting or terminating for any arrow), together with all arrows that contain it;

b) identification of the ends of an arrow α, together with rejection of α and any unnecessary arrow β (i.e., such that is equal to a path $\gamma = \gamma_1 \ldots \gamma_s$, where $\gamma_i \neq \beta$).

Note that Q is considered as a quiver with relations induced by the relations of commutativity (Q is not necessarily commutative).

2. The main result

We study connection between inj-finiteness of a finite poset and positive definiteness of its Tits form.

A poset A is said to be quasi-primitive if $\overrightarrow{A}$ is a disjoint union of chains (in the case when all arrows of every chain have the same direction, the poset is called primitive).

The main result of this paper is the following theorem.

Theorem 2. *Let A be a quasi-primitive poset which is not self-dual. Then both A and A^{op} are of inj-finite type if and only if the Tits form of A is positive.*

Proof. Sufficiency. Let the Tits form of A is positive. Then by Theorem 4 [7] A is, up to isomorphism and antiisomorphism, a subposet (proper or not) of one of the following posets: **1)** $1 \prec 2 \prec 7, 3 \prec 4 \prec 5 \prec 6 \prec 7$; **2)** $2 \prec 3, 4 \prec 5 \prec 6 \prec 7$; **3)** $2 \prec 7, 3 \prec 4 \prec 5 \prec 6 \prec 7$; **4)** $2 \prec 3, 2 \prec 7, 4 \prec 5 \prec 6 \prec 7$; **5)** $2 \prec 3 \prec 4, 2 \prec 7, 5 \prec 6 \prec 7$; **6)** $1 \prec 2 \prec 3, 4 \prec 7, 5 \prec 6 \prec 7$; **7)** $1 \prec 2 \prec 3, 4 \prec 5, 4 \prec 7, 6 \prec 7$; **8)** $1 \prec 3, 2 \prec 3, 2 \prec 7, 4 \prec 5 \prec 6 \prec 7$; **9)** $1 \prec 4, 2 \prec 3 \prec 4, 2 \prec 5 \prec 6 \prec 7$; **10)** $1 \prec 4, 2 \prec 3 \prec 4, 2 \prec 7, 5 \prec 6 \prec 7$; **11)** $1 \prec 5, 2 \prec 3 \prec 4 \prec 5, 2 \prec 7, 6 \prec 7$; **12)** $1 \prec 6, 2 \prec 3 \prec 4 \prec 5 \prec 6, 2 \prec 7$; **13)** $1 \prec 2, 1 \prec 4, 3 \prec 4, 3 \prec 7, 5 \prec 6 \prec 7$; **14)** $1 \prec 2, 1 \prec 5, 3 \prec 4 \prec 5, 3 \prec 7, 6 \prec 7$; **15)** $1 \prec 2 \prec 5, 3 \prec 4 \prec 5, 3 \prec 6 \prec 7$; **16)** $1 \prec 2 \prec \ldots \prec p, p+1 \prec p+2 \prec \ldots \prec p+q, 1 \prec p+q$ (the posets 1)–15) and 16) contain of elements 1,2,. . . , and are of order 7 and $p+q$, respectively). By Theorem 1 the poset A is of *inj*-finite type (even when it is self-dual).

Necessity. Let the Tits form of A is not positive. We prove that A or A^{op} is not of *inv*-finite type. By Theorem 3 [7] A contains (up to isomorphism and antiisomorphism) one of the following posets: **1)** $1 \prec 2 \prec 3 \prec 7, 4 \prec 5 \prec 6 \prec 7$; **2)** $1 \prec 2 \prec 8, 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8$; **3)** $1 \prec 2, 3 \prec 4, 5 \prec 6$; **4)** $2 \prec 3 \prec 6, 4 \prec 5 \prec 6$; **5)** $2 \prec 3 \prec 4, 5 \prec 6 \prec 7$; Y 7; **6)** $1 \prec 2, 3 \prec 7, 4 \prec 5 \prec 6 \prec 7$; **7)** $1 \prec 2 \prec 4, 3 \prec 4, 3 \prec 7, 5 \prec 6 \prec 7$; **8)** $2 \prec 3, 4 \prec 5 \prec 6 \prec 7 \prec 8$; **9)** $2 \prec 8, 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8$; **10)** $2 \prec 3, 2 \prec 8, 4 \prec 5 \prec 6 \prec 7 \prec 8$; **11)** $1 \prec 2 \prec 3 \prec 4, 5 \prec 8, 6 \prec 7 \prec 8$; **12)** $1 \prec 2 \prec 3 \prec 4, 5 \prec 6, 5 \prec 8, 7 \prec 8$; **13)** $1 \prec 3, 2 \prec 3, 2 \prec 8, 4 \prec 5 \prec 6 \prec 7 \prec 8$; **14)** $1 \prec 4, 2 \prec 3 \prec 4, 2 \prec 5 \prec 6 \prec 7 \prec 8$; **15)** $1 \prec 4, 2 \prec 3 \prec 4, 2 \prec 8, 5 \prec 6 \prec 7 \prec 8$; **16)** $1 \prec 7, 2 \prec 3 \prec 4 \prec 5 \prec 6 \prec 7, 2 \prec 8$; **17)** the four elements $1, 2, 3, 4$ are pairwise incomparable (the posets 1)–16) consist of elements $1, 2, \ldots, p$, where in each case p is a maximal indicated number). By Theorem 1 each of the posets $1)$–$13), 14^{\mathrm{op}}), 15), 17)$ are not of *inj*-finite type (even when it is self-dual). The poset $P = 16)$ is self-dual and of *inj*-finite type. But if A contains P, then it contains the poset $P \cup 9$, where either 9 is incomparable to any $i \in P$, or 9 is incomparable to any $i \in P \setminus 8$ and $9 \prec 8$, or 9 is incomparable to any $i \in P \setminus \{2, 8\}$ and $9 \succ 8$ (the case $A = P$ is impossible since A is not self-dual); in the first case P contains the poset 17), and in the second and third ones P is not of *inj*-finite type by Theorem 1.

Thus A or A^{op} is not of *inv*-finite type.

Theorem 2 is proved.

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