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Fredholm boundary-value problems for linear delay systems defined by pairwise permutable matrices

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Abstract. The paper deals with a Fredholm boundary value problem for a linear delay system with several delays defined by pairwise permutable constant matrices. The initial value condition is given on a finite interval and the boundary condition is given by a linear vector functional. A sufficient condition for the existence of solutions of this type of boundary value problem is proved. Moreover, a family of linearly independent solutions in an explicit general *analytic* form is constructed under the assumption that the number of boundary conditions (determined by the dimension of linear vector functional) do not coincide with the number of unknowns of the system of the delay differential equations. The proof of this result is based on a representation of solutions by using the so-called multi-delayed matrix exponential and a method of a pseudo-inverse matrix of the Moore–Penrose type.

Keywords: boundary-value problem, multi-delayed system, Moore–Penrose pseudo-inverse matrix.

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1 Introduction

The aim of the paper is to prove an existence result for the following boundary-value problem:

$$\begin{aligned} &\dot{z}(t) = Az(t) + B_1 z(t-\tau_1) + \cdots + B_n z(t-\tau_n) + g(t), \qquad t \in [0,b], \\ &z(s) = \psi(s), \quad \text{if } \ s \in [-\tau, 0], \end{aligned} \tag{1.1}$$

$$lz(\cdot) = \alpha \in \mathbb{R}^m, \tag{1.2}$$

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where $\tau_1, \ldots, \tau_n > 0$, $(n > 0)$, $\tau := \max\{\tau_1, \ldots, \tau_n\}$ and A, $B_1, \ldots, B_n$ are $N \times N$ constant permutable matrices such that $AB_i = B_iA$, $B_iB_j = B_jB_i$ for each $i, j \in \{1, \ldots, n\}$ and $g(t)$ is an N-dimensional column-vector, with components in the space $L_p[0,b]$ $(1 < p < \infty)$ being functions integrable on $[0,b]$; $\psi\colon \mathbb{R} \setminus [0,b] \to \mathbb{R}^N$ is a given N-dimensional column-vector function; α is an m-dimensional constant vector-column, l is an m-dimensional linear vector-functional, defined on the space $D_p[0,b]$ of n-dimensional vector-functions absolutely continuous on $[0,b]$: $l = \operatorname{col}(l_1, \ldots, l_m)\colon D_p[0,b] \to \mathbb{R}^m$, $l_i\colon D_p[0,b] \to \mathbb{R}$. It is not very difficult to prove that in this space such problems for functional-differential equations are of Fredholm's type with nonzero index (see, e.g., [1, 4, 5]).

First of all we consider initial value problems for a system of linear differential equations with delays defined by pairwise permutable matrices:

$$\begin{aligned} &\dot{z}(t) = Az(t) + B_1z(t-\tau_1) + \cdots + B_nz(t-\tau_n) + g(t), \qquad t \in [0,b], \\ &z(s) = \psi(s), \quad \text{if } s \in [-\tau, 0]. \end{aligned} \tag{1.3}$$

Using the notations

$$(S_{h_i}z)(t) := \begin{cases} z(h_i(t)) & \text{if } h_i(t) := t - \tau_i \in [0,b], \\ 0 & \text{if } h_i(t) := t - \tau_i \notin [0,b], \end{cases} \tag{1.4}$$

$$\psi^{h_i}(t) := \begin{cases} 0 & \text{if } h_i(t) \in [0,b], \\ \psi(h_i(t)) & \text{if } h_i(t) \notin [0,b], \end{cases} \tag{1.5}$$

it is possible to rewrite initial value problems for (1.3) as an operator equation

$$(Lz)(t) := \dot{z}(t) - Az(t) - \sum_{i=1}^{n} B_i(S_{h_i}z)(t) = \varphi(t), \tag{1.6}$$

where $(S_{h_i}z)(t)$ is an N-dimensional column-vector and $\varphi(t)$ is an N-dimensional column-vector defined by the formula

$$\varphi(t) := g(t) + \sum_{i=1}^{n} B_i\psi^{h_i}(t) \ \in \ L_p[0,b].$$

The operator $S_{h_i}\colon D_p \to L_p$ admits the following representation:

$$(S_{h_i}z)(t) = \int_0^b \chi_{h_i}(t,s)\dot{z}(s)\,ds + \chi_{h_i}(t,0)z(0),$$

where $\chi_{h_i}(t,s)$ is the characteristic function of the set

$$\Omega = \{(t,s) \in [0,b] \times [0,b] : 0 \le s \le h_i(t) \le b\},$$

defined by

$$\chi_{h_i}(t,s) = \begin{cases} 1, & (t,s) \in \Omega, \\ 0, & (t,s) \notin \Omega. \end{cases}$$

We will investigate the equation (1.6) assuming that the operator L maps a Banach space $D_p[0,b]$ of absolutely continuous functions $z\colon [0,b] \to \mathbb{R}^N$ with the norm

$$\|z(t)\|_{D_p} = \|\dot{z}(t)\|_{L_p} + \|z(0)\|_{\mathbb{R}^N}$$

into the Banach space $L_p[0,b]$ $(1 < p < \infty)$ of functions $\varphi\colon [0,b] \to \mathbb{R}^N$ integrable on $[0,b]$, equipped with the standard norms for these spaces. It is well-known [1] that, in the considered spaces, problem (1.6) is equivalent to initial value problem (1.3). The transformations (1.4), (1.5) allow to add the initial function $\psi(s)$, $s < 0$ to an inhomogeneity and thus to generate an additive and homogeneous operation not depending on ψ, and without a classical assumption regarding the continuous connection of solution $z(t)$ with the initial function $\psi(t)$ at the point $t = 0$. *A solution of differential system* (1.6) *is defined as a vector-function* $z(t) \in D_p[0,b]$ *absolutely continuous on* $[0,b]$ *with* $\dot{z}(t) \in L_p[0,b]$, *if it satisfies the system* (1.6) *almost everywhere on* $[0,b]$. Such a treatment makes it possible to apply to the equation (1.6) with the linear and bounded operator L well developed methods of linear functional analysis. It is well-known (see, e.g., [1, 3, 4]) that an inhomogeneous operator equation (1.6) with delayed arguments is solvable for an arbitrary right-hand side $\varphi(t) \in L_p[0,b]$ and has an N-dimensional family of solutions ($\dim\ker L = N$) in the form

$$z(t) = X(t)c + \int_0^b K(t,s)\varphi(s)\,ds, \quad \text{for all } c \in \mathbb{R}^N \tag{1.7}$$

where the kernel $K(t,s)$ of the integral is an $(N \times N)$-dimensional Cauchy matrix $K(t,s)$ being, for every fixed s, a solution of the matrix Cauchy problem:

$$(LK(\,\cdot\,,s))(t) := \frac{\partial K(t,s)}{\partial t} - AK(t,s) - \sum_{i=1}^{n} B_i(S_{h_i}K(\cdot,s))(t) = 0, \qquad K(s,s) = I.$$

In the following we assume that the matrix $K(t,s)$ is defined in the square $[0,b]\times[0,b]$ and $K(t,s) \equiv 0$ if $0 \le t < s \le b$. A fundamental $(n \times n)$-dimensional matrix for the homogeneous $(\varphi(t) \equiv 0)$ equation (1.6) has the form $X(t) = K(t,0)$ (see [1]).

A disadvantage of this approach, when investigating the above-formulated problem, is the necessity to find analytically a fundamental $X(t)$ and the Cauchy $K(t,s)$ matrices [5, 7]. Below we consider the case of a system with delays, when this problem can be directly solved. In this case the problem of how to construct the Cauchy matrix is successfully solved *analytically* due to a delayed matrix exponential defined in [6] and generalized to the case of several delays in [8].

2 Multi-delay matrix exponential

We recall the definition of the multi-delay matrix exponential defined in [8].

Definition 2.1. Let $B_1, \ldots, B_n$ be pairwise permutable $N \times N$ matrices, i.e., $B_iB_j = B_jB_i$ for each $i, j \in \{1, \ldots, n\}$. For each $j = 2, \ldots, n$ we define $N \times N$ multi-delayed matrix exponential corresponding to delays $\tau_j > 0$ and matrices $B_1, \ldots, B_j$ as follows

$$e_{\tau_1,\ldots,\tau_j}^{B_1,\ldots,B_j t} := \begin{cases} \Theta, & \text{if } t < -\tau_j, \\ X_{j-1}(t+\tau_j), & \text{if } \tau_j \le t < 0, \\ \vdots & \\ X_{j-1}(t+\tau_j) + B_j \int_0^t X_{j-1}(t-s_1)X_{j-1}(s_1)\,ds_1 & \\ + \cdots + B_j^k \int_{(k-1)\tau_j}^t \int_{(k-1)\tau_j}^{s_1} \cdots \int_{(k-1)\tau_j}^{s_{k-1}} X_{j-1}(t-s_1) & \\ \times \prod_{i=1}^{k-1} X_{j-1}(s_i - s_{i+1})X_{j-1}(s_k - (k-1)\tau_j)\,ds_k \ldots ds_1, & \text{if } (k-1)\tau_j \le t < k\tau_j, \\ & k = 1, 2, \ldots, \end{cases} \tag{2.1}$$

where: $X_{j-1}(t) = e_{\tau_1,\dots,\tau_{j-1}}^{B_1,\dots,B_{j-1}(t-\tau_{j-1})}$, Θ is the null $N \times N$ matrix, function $e_{\tau_1,\dots,\tau_n}^{B_1,\dots,B_n t}$ has the properties well described in [8, Lemma 7].

Using the multi-delayed matrix exponential (2.1) we can represent a solution $z(t)$ of a corresponding linear system (1.6) with multiple delays and pairwise permutable matrices in the form (1.7), where

$$K(t,s) := Y(t-s) \quad \text{if } 0 \le s \le t \le b, \qquad K(t,s) \equiv 0 \quad \text{if } 0 \le t < s \le b \tag{2.2}$$

and

$$Y(t) = \mathrm{e}^{At}\, \mathrm{e}_{\tau_1,\dots,\tau_n}^{\widetilde{B}_1,\dots,\widetilde{B}_n(t-\tau_n)}, \qquad \widetilde{B}_i = \mathrm{e}^{-A\tau_i} B_i, \qquad i = 1,\dots,n,$$

$$X(t) := K(t,0) = Y(t) = \left[\mathrm{e}^{At}\, \mathrm{e}_{\tau_1,\dots,\tau_n}^{\widetilde{B}_1,\dots,\widetilde{B}_n(t-\tau_n)}\right].$$

3 Fredholm boundary-value problem

Using the results [3, 4], it is easy to derive results for a general boundary-value problem if the number m of boundary conditions does not coincide with the number N of unknowns in a differential system with a delay. We derive such results in an *explicit analytical* form. We consider the boundary-value problem

$$\dot z(t) - Az(t) - \sum_{i=1}^{n} B_i (S_{h_i} z)(t) = \varphi(t), \qquad t \in [0,b], \tag{3.1}$$

$$lz(\cdot) = \alpha \in \mathbb{R}^m, \tag{3.2}$$

where α is an m-dimensional constant vector-column, $l = \operatorname{col}(l_1,\dots,l_m)\colon\ D_p[0,b] \to \mathbb{R}^m$, $(l_i\colon D_p[0,b] \to \mathbb{R})$ is an m-dimensional linear vector-functional defined on the space $D_p[0,b]$ of N-dimensional vector-functions absolutely continuous on $[0,b]$. As above, we state that, in the spaces considered, this problem is equivalent to problem (1.1), (1.2), where

$$\varphi(t) := g(t) + \sum_{i=1}^{n} B_i \psi^{h_i}(t) \ \in\ L_p[0,b].$$

We will derive sufficient and necessary conditions, and a representation of the solutions $z \in D_p[0,b]$, $\dot z(t) \in L_p[0,b]$ of the boundary-value problem (3.1), (3.2).

Substituting the general solution (1.7) of the equation (3.1) into the boundary condition (3.2), in accordance with (2.2), we will have the algebraic system

$$Qc = \alpha - l \int_0^b K(\cdot\,,s)\varphi(s)\, ds \tag{3.3}$$

with the constant $m \times N$ dimension matrix

$$Q = lX(\cdot) = l\left[\mathrm{e}^{A\cdot}\, \mathrm{e}_{\tau_1,\dots,\tau_n}^{\widetilde{B}_1,\dots,\widetilde{B}_n(\cdot-\tau_n)}\right].$$

Preserving the above used notation [4], we have: $\operatorname{rank} Q = n_1 \le \min(m,N)$, $P_Q := I_N - Q^+Q$ is an $N \times N$-dimensional matrix (orthogonal projection) projecting the space $\mathbb{R}^N$ to the kernel $(\ker Q)$ of the matrix Q, $P_{Q^*} := I_m - QQ^+$ in an $m \times m$-dimensional matrix (orthogonal projection) projecting the space $\mathbb{R}^m$ to the kernel Q^* of the transposed matrix $Q^* = Q^T$. Using the property

$$\operatorname{rank} P_{Q^*} = m - \operatorname{rank} Q^* = d = m - n_1$$

we will denote by $P_{Q_d^*}$ a $d \times m$-dimensional matrix constructed from d linearly independent rows of the matrix P_{Q^*}. Using the property

$$\operatorname{rank} P_Q = N - \operatorname{rank} Q = r = N - n_1$$

we will denote by P_{Q_r} an $N \times r$-dimensional matrix constructed from r linearly independent columns of the matrix P_Q.

Then (see [4, p. 79]) the condition

$$P_{Q_d^*}\left\{\alpha - l\int_0^b K(\cdot,s)\varphi(s)\,ds\right\} = 0 \tag{3.4}$$

is necessary and sufficient for algebraic system (3.3) to be solvable and if such condition is true, system (3.3) has a solution

$$c = P_{Q_r}c_r + Q^+\left\{\alpha - l\int_0^b K(\cdot,s)\varphi(s)\,ds\right\} \quad \text{for all } c_r \in \mathbb{R}^r, \tag{3.5}$$

where Q^+ is an $N \times m$-dimensional matrix pseudo-inverse with respect to the $m \times N$-dimensional matrix Q.

Substituting the constant $c \in \mathbb{R}^N$ defined by (3.5) into (1.7), we get a formula for the general solution of problem (3.1), (3.2):

$$z(t,c_r) = X_r(t)c_r + (G\varphi)(t) + X(t)Q^+\alpha, \tag{3.6}$$

where $X_r(t) = X(t)P_{Q_r}$,

$$(G\varphi)(t) := \int_0^b G(t,s)\varphi(s)\,ds$$

is a generalized Green operator, and

$$G(t,s) := K(t,s) - X(t)Q^+ lK(\,\cdot\,,s)$$

is a generalized Green matrix, corresponding to the boundary-value problem (3.1), (3.2). Therefore, the following theorem holds.

Theorem 3.1. *If* $\operatorname{rank} Q = n_1 \le \min(m,N)$, *then the homogeneous problem corresponding to problem* (3.1), (3.2) *(with* $\varphi(t) = 0, \alpha = 0$*) has exactly* r *(where* $r = N - n_1$*) linearly independent solutions in the space* $D_p[0,b]$. *The inhomogeneous problem* (3.1), (3.2) *is solvable in the space* $D_p[0,b]$ *if and only if* $\varphi(t) \in L_p[0,b]$ *and* $\alpha \in \mathbb{R}^m$ *satisfy* d *linearly independent conditions* (3.4). *Then it has an* r*-dimensional family of linearly independent solutions* $z(t,c_r) : z(\cdot,c_r) \in D_p[0,b]\,,\ \dot z(\cdot,c_r) \in L_p[0,b]$, *represented in an explicit form* (3.6).

The case of $\operatorname{rank} Q = N$ implies the inequality $m \ge N$, i.e., the boundary-value problem is overdetermined, the number of boundary conditions is not less than the number of unknowns, Theorem 3.1 has the following corollary.

Corollary 3.2. *If* $\operatorname{rank} Q = N$, *then the homogeneous problem has only the trivial solution. Inhomogeneous problem* (3.1), (3.2) *is solvable if and only if*

$$P_{Q_d^*}\left\{\alpha - l\int_0^b K(\,\cdot\,,s)\varphi(s)\,ds\right\} = 0$$

where $d = m - N$. *Then the unique solution can be represented as*

$$z(t) = (G\varphi)(t) + X(t)Q^+\alpha.$$

The case of $\operatorname{rank} Q = m$ is interesting as well. Then the inequality $m \le N$ holds, i.e., the boundary-value problem is underdetermined. In this case, Theorem 3.1 has the following corollary.

Corollary 3.3. *If* $\operatorname{rank} Q = m$, *then the boundary-value problem has an* r*-dimensional* ($r = N - m$) *family of solutions. The inhomogeneous problem* (3.1), (3.2) *is solvable for arbitrary* $\varphi(t) \in L_p[0, b]$ *and* $\alpha \in \mathbb{R}^m$ *and has an* r*-parametric family of solutions*

$$z(t, c_r) = X_r(t)c_r + (G\varphi)(t) + X(t)Q^+\alpha.$$

Finally, combining both particular cases mentioned above, we get the following.

Corollary 3.4. *If* $\operatorname{rank} Q = N = m$, *then the homogeneous problem has only the trivial solution. The inhomogeneous boundary-value problem* (3.1), (3.2) *is solvable for arbitrary* $\varphi(t) \in L_p[0, b]$ *and* $\alpha \in \mathbb{R}^N$, *and has a unique solution*

$$z(t) = (G\varphi)(t) + X(t)Q^{-1}\alpha.$$

Corollary 3.5. *If* $A = 0$ *and* $i = 1$, *then from Theorem 3.1 we obtain the result published in [2].*

4 Example

Consider the boundary value problem with two delays [8, p. 3350]

$$\dot z(t) = b_1 z\left(t - \frac{3}{4}\right) + b_2 z(t-1) + \varphi(t), \qquad t \in [0, 1], \tag{4.1}$$

$$\ell z(\cdot) = \alpha, \tag{4.2}$$

where $\ell = \operatorname{col}(l_1, l_2)$ is a two-dimensional vector functional:

$$l_1 z(\cdot) := -\frac{b_1}{2}\, z(0) + 2z(1),$$
$$l_2 z(\cdot) := \left(2 + \frac{b_1}{4}\right) z(0) - z(1),$$

$\alpha = \operatorname{col}(\alpha_1, \alpha_2) \in \mathbb{R}^2$.

The general solution of the equation (4.1) has the form

$$z(t) = Y(t)c + \int_0^t Y(t-s)\varphi(s)\, ds, \tag{4.3}$$

where $Y(t)$ is the solution of the corresponding homogeneous (4.1) equation on the interval $[0, 1]$ [8, p. 3351]

$$Y(t) = e^{b_1, b_2(t-\tau_2)}_{\tau_1, \tau_2} = \begin{cases} 0, & t < 0, \\ 1, & 0 \le t < \frac{3}{4}, \\ 1 + b_1(t - \frac{3}{4}), & \frac{3}{4} \le t \le 1. \end{cases}$$

Substituting the general solution (4.3) into the boundary conditions (4.2), we obtain an algebraic equation

$$Qc = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} - \begin{bmatrix} 2\int\limits_0^1 Y(1-s)\varphi(s)\, ds \\ -\int\limits_0^1 Y(1-s)\varphi(s)\, ds \end{bmatrix}. \tag{4.4}$$

For boundary value problem (4.1), (4.2) the matrix Q has the form

$$Q = \ell Y(\cdot) = \begin{bmatrix} -\frac{b_1}{2}\, Y(0) + 2Y(1) \\ (2 + \frac{b_1}{4})\, Y(0) - Y(1) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Then

$$P_Q = 0, \qquad P_{Q^*} = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix}, \qquad P_{Q_d^*} = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \end{bmatrix}, \qquad Q^+ = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \end{bmatrix}.$$

The equation (4.4), and hence the boundary value problem (4.1), (4.2) is solvable if and only if condition

$$P_{Q_d^*} \left\{ \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} - \begin{bmatrix} 2\int_0^1 Y(1-s)\varphi(s)\, ds \\ -\int_0^1 Y(1-s)\varphi(s)\, ds \end{bmatrix} \right\} = 0$$

is satisfied, and after the transformation that is of the form

$$\alpha_1 - 2\alpha_2 - 4\int_0^1 Y(1-s)\varphi(s)\, ds = 0, \tag{4.5}$$

where

$$Y(1-s) = \begin{cases} 1, & 0 \le 1-s \le \frac{3}{4} \Leftrightarrow \frac{1}{4} \le s \le 1, \\ 1 + b_1\left(\frac{1}{4} - s\right), & \frac{3}{4} \le 1-s \le 1 \Leftrightarrow 0 \le s \le \frac{1}{4}. \end{cases}$$

Since $P_Q = 0$, then under the condition (4.5), equation (4.4) has a unique solution

$$c = Q^+ \left\{ \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} - \begin{bmatrix} 2\int_0^1 Y(1-s)\varphi(s)\, ds \\ -\int_0^1 Y(1-s)\varphi(s)\, ds \end{bmatrix} \right\}. \tag{4.6}$$

Substituting c from (4.6) in the formula (4.3) we have a unique solution of the boundary value problem (4.1), (4.2)

$$z(t) = \int_0^t Y(t-s)\varphi(s)\, ds - Y(t)Q^+ \begin{bmatrix} 2\int_0^1 Y(1-s)\varphi(s)\, ds \\ -\int_0^1 Y(1-s)\varphi(s)\, ds \end{bmatrix} + Y(t)Q^+ \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix},$$

which after conversion has the form

$$z(t) = \int_0^t Y(t-s)\varphi(s)\, ds - \frac{3}{5}\, Y(t) \int_0^1 Y(1-s)\varphi(s)\, ds + Y(t)\left[\frac{2\alpha_1}{5} + \frac{\alpha_2}{5}\right]$$

and the generalized Green matrix, corresponding to the boundary-value problem (4.1), (4.2), has the form

$$G(t,s) = \begin{cases} Y(t-s) - \frac{3}{5} Y(t)Y(1-s), & 0 \le s \le t, \\ -\frac{3}{5} Y(t)Y(1-s), & t < s \le 1. \end{cases} \tag{4.7}$$

For example, the condition (4.5) will be fulfilled for the inhomogeneities of the following form:

$$\varphi(t) = t, \qquad \alpha_1 = 2, \qquad \alpha_2 = -\frac{b_1}{4^3 \cdot 3}.$$

On the interval $0 \leq t < \frac{3}{4}$, we have in the Green matrix (4.7) $Y(t) = 1$, $Y(t-s) = 1$ and the solution of the boundary value problem (4.1), (4.2) for

$$\varphi(t) = t, \qquad \alpha_1 = 2, \qquad \alpha_2 = -\frac{b_1}{4^3 \cdot 3}$$

will have the form

$$z_1(t) = \frac{t^2}{2} - \frac{3}{5}\left[\frac{1}{2} + \frac{b_1}{4^3 \cdot 6}\right] + \frac{2}{5}\alpha_1 + \frac{1}{5}\alpha_2.$$

On the interval $\frac{3}{4} \leq t \leq 1$ we have in the Green matrix (4.7)

$$Y(t) = 1 + b_1\left(t - \frac{3}{4}\right),$$
$$Y(t-s) = 1 + b_1\left(t - \frac{3}{4} - s\right)$$

and the solution of the boundary value problem (4.1), (4.2) will have the form

$$z_2(t) = \frac{t^2}{2} + b_1 t\frac{(t-\frac{3}{4})^2}{2} - b_1\frac{t^3}{3} + b_1\frac{3t^2}{8} - b_1\frac{9}{4^3 \cdot 2} - \frac{3}{5}\left[1 + b_1\left(t - \frac{3}{4}\right)\right]\left[\frac{1}{2} + \frac{b_1}{4^3 \cdot 6}\right] + \left[1 + b_1\left(t - \frac{3}{4}\right)\right]\left[\frac{2}{5}\alpha_1 + \frac{1}{5}\alpha_2\right].$$

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