Algebra and Discrete Mathematics
Number 2. **(2005).** pp. 20 – 35

On posets of width two with positive Tits form

Vitalij M. Bondarenko, Marina V. Styopochkina

Abstract. We give a complete description of the finite posets of width two with the Tits form to be positive. This problem arises in studying the categories of representations of posets of finite type.

The quadratic Tits form, introduced by P. Gabriel [1] for quivers, S. Brenner [2] for quivers with relations and Yu. A. Drozd [3] for posets plays an important role in representation theory (see also definitions of the Tits form for wide classes of matrix problems without relations in [4] and [5]). In particular, there are many results on connections between representation types of various objects and properties of the Tits forms. The reader interested in this theme is referred to the monographs [6], [7] and, e.g., the paper of [8]–[19] (with the bibliographies therein). Above all one must mention the well known results that a quiver (resp. a poset) is of finite type if and only if its Tits form is positive (resp. weakly positive); see [1] and [2], respectively. Our paper is devoted to study the structure of finite posets with positive Tits form which arise in studying the categories of representations of posets of finite type.

1. Formulation of the main result

Throughout the paper, all posets (partially ordered sets) are finite. In considering a poset $S = (A, \leqslant)$ the set A will not be written and therefore we keep to the following conventions: by a subset of S we mean a subset of A together with the induced order relation (which is denoted by the same symbol $\leqslant$), we write $x \in S$ instead of $x \in A$, etc.

Let S be a poset. Given nonempty subsets $X, Y \in S$, we write $X \lhd Y$ if $x < y$ for some $x \in X$, $y \in Y$, and $X \ntriangleleft Y$ if otherwise. We say that

2000 Mathematics Subject Classification: *15A63, 16G20, 16G60.*

Key words and phrases: *minimax (semiminimax) sum,* (min, max)*-equivalent posets, positive form, the quadratic Tits form.*

S is a *sum* of subposets A and B (and write $S = A + B$) if $A \cap B = \varnothing$ and $S = A \cup B$. If any two elements $a \in A$ and $b \in B$ are incomparable, this sum is called *direct*; one denotes such sum by $S = A \coprod B$. The sum $S = A + B$ is said to be an *one-sided* if $B \ntriangleleft A$ or $A \ntriangleleft B$. When the first (resp. second) case occurs, we say that the sum is *right* (resp. *left*); note that a direct sum is both right and left. Finally, the sum $S = A + B$ is said to be *minimax* (resp. semiminimax) if $x < y$ with x and y belonging to different summands implies that x is minimal and (resp. or) y maximal in S. [1]

For a sum S of posets A and B let $R^{<}(A, B)$ denotes the set of pairs $(x, y) \in A \times B$ with $x < y$. Such a pair (a, b) is said to be *short* if there is no other such a pair (a', b') satisfying $a \leqslant a'$, $b' \leqslant b$. By $R_0^{<}(A, B)$ we denote the subset of all short pairs from $R^{<}(A, B)$. We call the order $r_0 = r_0(A, B)$ of

$$R_0(A, B) = R_0^{<}(A, B) \cup R_0^{<}(B, A)$$

the *rank* of the sum S.

Let S be a poset and $\mathbb{Z}^{S \cup 0}$ the cartesian product of $|S| + 1$ copies of $\mathbb{Z}$ (consisting of all vectors $z = (z_i), i \in S \cup 0$), where $\mathbb{Z}$ denotes the integer numbers. The *quadratic Tits form* of S is by definition the form $q_S : \mathbb{Z}^{S \cup 0} \to \mathbb{Z}$ defined by the equality

$$q_S(z) = z_0^2 + \sum_{i \in S} z_i^2 + \sum_{\substack{i<j, \\ i,j \in S}} z_i z_j - z_0 \sum_{i \in S} z_i.$$

This form (as an arbitrary one) is called *positive* if it takes a positive value on every nonzero vector $z \in \mathbb{Z}^{S \cup 0}$, and nonpositive if otherwise.

Recall that a linear ordered set is also called a *chain*. A poset with the only pair of incomparable elements will be called an *almost chain*. The *width* of a poset S is defined to be the maximum number of pairwise incomparable elements of S and is denoted by $w(S)$.

If S is a chain, then its Tits form is positive (see, for instance, [20], or Section 3).

Our aim in this paper is to classifying the posets of width 2 with positive Tits form.

In the case when the order of a poset is at least 8 we have the following theorem.

[1]The notion of a sum have been introduced by the internal way. The external way is possible but not convenient, however it is natural in some special cases, and in particular when one has a right (left) minimax sum of two chains.

Theorem 1. Let S be a poset of width 2 and order at least 8. Then the Tits form of S is positive if and only if one of the following condition holds:

1) S is a direct sum of two chains;
2) S is an almost chain;
3) S is a one-sided minimax sum of rank 1 of two chains.

From the main result of [18] it follows the above theorem but for a poset of sufficiently larger order. By Theorem 1 the number of the remaining possibilities for orders of posets with the Tits form being positive becomes explicitly known to us (and is not large). A complete list of the posets of width 2 and order smaller than 8 with positive Tits form will be given in Section 4; see Proposition 3 and Theorem 2. Note that this list is used in the proof of Theorem 1.

We also give a complete list of critical posets (of width 2) with respect to positivity of the Tits form (see Theorem 3).

2. Subsidiary statements

2.1. Posets with nonpositive Tits form

In this subsection we indicate some posets with the Tits form to be nonpositive. We will see in the next sections that these posets are minimal ones with nonpositive Tits form; moreover, they form (up to some natural isomorphisms) the full set of such minimal posets.

When we determine some poset, the corresponding order relation is given up to transitivity. In the case when the elements of a poset are denoted by integer numbers, the order relation is denoted by $\prec$ (to distinguish between the given relation and the natural ordering of the integer numbers). An element of a poset S is said to be *nodal* if it is comparable to any other element.

Lemma 1. The Tits form is nonpositive for the following posets of order smaller than 8 and of width 2 :

$$\begin{aligned}
T_1 &= \{1 \prec 3, 1 \prec 4, 2 \prec 3, 2 \prec 4\},\\
T_2 &= \{1 \prec 2 \prec 3, 4 \prec 5 \prec 6, 2 \prec 5\},\\
T_3 &= \{1 \prec 2, 3 \prec 4 \prec 5 \prec 6, 2 \prec 5\},\\
T_4 &= \{2 \prec 3 \prec 4 \prec 5 \prec 6 \prec 7, 1 \prec 5\},\\
T_5 &= \{1 \prec 2, 3 \prec 4 \prec 5 \prec 6 \prec 7, 1 \prec 5\},\\
T_6 &= \{1 \prec 2 \prec 3, 4 \prec 5 \prec 6 \prec 7, 1 \prec 5\},\\
T_7 &= \{1 \prec 2 \prec 3, 4 \prec 5 \prec 6 \prec 7, 3 \prec 7\}.
\end{aligned}$$

(T_i consists of elements $1, 2, \ldots, n$, where n is the greatest number in its definition.)

Lemma 2. The Tits form is nonpositive for the following posets of order 8 and of width 2 with nodal elements :

$$
\begin{aligned}
T_8 &= \{2 \prec 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8, 1 \prec 7\},\\
T_9 &= \{2 \prec 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8, 1 \prec 4\},\\
T_{10} &= \{1 \prec 2, 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8, 2 \prec 8\},\\
T_{11} &= \{1 \prec 2, 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8, 1 \prec 4, 2 \prec 8\},\\
T_{12} &= \{1 \prec 2, 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8, 1 \prec 7, 2 \prec 8\},\\
T_{13} &= \{1 \prec 2, 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8, 1 \prec 4, 2 \prec 5\}.
\end{aligned}
$$

Lemma 3. The Tits form is nonpositive for the following posets of order 8 and of width 2 without nodal elements :

$$
\begin{aligned}
T_{14} &= \{1 \prec 2, 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8, 1 \prec 7\},\\
T_{15} &= \{1 \prec 2, 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8, 1 \prec 4\},\\
T_{16} &= \{1 \prec 2 \prec 3, 4 \prec 5 \prec 6 \prec 7 \prec 8, 2 \prec 8\},\\
T_{17} &= \{1 \prec 2 \prec 3, 4 \prec 5 \prec 6 \prec 7 \prec 8, 1 \prec 7\},\\
T_{18} &= \{1 \prec 2 \prec 3, 4 \prec 5 \prec 6 \prec 7 \prec 8, 1 \prec 7, 2 \prec 8\},\\
T_{19} &= \{1 \prec 2 \prec 3 \prec 4, 5 \prec 6 \prec 7 \prec 8, 1 \prec 7\},\\
T_{20} &= \{1 \prec 2 \prec 3 \prec 4, 5 \prec 6 \prec 7 \prec 8, 1 \prec 7, 2 \prec 8\}.
\end{aligned}
$$

Proof of Lemmas 1–3. The Tits form of $T = T_s$ ($s = 1, 2 \ldots, 20$) is denoted by $q_s(z)$. The coordinates z_i of a vector $z \in \mathbb{Z}^{T \cup 0}$ will be arranged in the natural way (in increasing order of the integer index $i \in T \cup 0$).

It is easy to verify that the following numbers are zero:

$$
\begin{aligned}
&q_1(0, 1, 1, -1, -1), && q_2(0, 1, 1, -1, 1, -1, -1),\\
&q_3(1, 1, 1, 1, 1, -1, -1), && q_4(1, 2, 1, 1, 1, -1, -1, -1),\\
&q_5(0, -2, 1, -1, -1, 1, 1, 1), && q_6(1, -2, 1, 1, -1, 1, 1, 1),\\
&q_7(2, 1, 1, 1, 1, 1, 1, -2), && q_8(2, 3, 1, 1, 1, 1, 1, -2, -2),\\
&q_9(1, 3, 2, 2, -1, -1, -1, -1, -1), && q_{10}(3, 2, 2, 1, 1, 1, 1, 1, -3),\\
&q_{11}(1, -2, 2, -1, 1, 1, 1, 1, -1), && q_{12}(2, 2, 1, 1, 1, 1, 1, -1, -2),\\
&q_{13}(1, 1, 2, 2, 1, -1, -1, -1, -1), && q_{14}(1, 3, -1, 1, 1, 1, 1, -2, -2),\\
&q_{15}(1, -3, 2, -2, 1, 1, 1, 1, 1), && q_{16}(2, 2, 2, -1, 1, 1, 1, 1, -3),\\
&q_{17}(0, 3, -1, -1, 1, 1, 1, -2, -2), && q_{18}(1, 2, 1, -1, 1, 1, 1, -1, -2),\\
&q_{19}(1, -3, 1, 1, 1, -1, -1, 2, 2), && q_{20}(0, 2, 1, -1, -1, 1, 1, -1, -2).
\end{aligned}
$$

Lemmas 1–3 are immediate from this. □

Below we will say "case i)" instead of "the case $T = T_i$" ($i = 1, 2, \ldots, 20$).

Since the Tits forms of a poset T and the dual poset T^{op} are the same, the dual lemmas, i.e. those with the dual posets T_1^{op}–T_{20}^{op} instead of T_1–T_{20}, are hold. We always assume that, for a poset T, $T^{\mathrm{op}} = T$ as usual sets (then $x < y$ in $T^{\mathrm{op}} = S$ iff $x > y$ in T).

2.2. Statements on the Tits form

For subsets X and Y of some poset, we write $X < Y$ if $x < y$ for any $x \in X, y \in Y$ (clearly, $X < Y$ when X or Y is empty). When S and S' are posets, we denote by $[S < S']$ the disjoint union $S \cup S'$ with the smallest order relation which contains those on S and S', and such that $S < S'$. Obviously, $[S < \varnothing] = S$ and $[\varnothing < S'] = S'$. Singletons $\{x\}$ are often identified with the elements x themselves.

Let S be a poset. Recall that an element of S is said to be nodal if it is comparable to any other element. The set of all nodal elements of S is denoted by S_0; obviously, S_0 is a chain. Set $S^\circ = S \setminus S_0$. For a given element x of S, we denote by $N_S(x)$ (or simply $N(x)$ if no confusion is possible) the subset of all elements $y \in S$ that are not comparable to x; obviously, x is a nodal element of the subset $S \setminus N_S(x)$. We say that posets S and S' are 0-*isomorphic* and write $S \cong_0 S'$ if $S \setminus S_0 \cong S' \setminus S'_0$ and $|S_0| = |S'_0|$ (the notation $T \cong T'$ means the usual isomorphism). Posets S and S' is said to be *antiisomorphic* (resp. 0-*antiisomorphic*) if S^{op} and S' is isomorphic (resp. 0-isomorphic).

From the definition of the Tits form it follows immediately the following lemma.

Lemma 4. Let S be a poset, and set $S^+ = [(S \setminus S_0) < S_0]$ and $S^- = [S_0 < (S \setminus S_0)]$. Then the Tits form of $S^\pm$ is the same as that of S.

The following lemma is a generalization of the previous one (and follows from the definition of the Tits form too).

Lemma 5. The Tits form of 0-isomorphic or 0-antiisomorphic posets are isomorphic (i.e. the same up to numbering of their variables).

Now we recall some definitions and assertions from [21].

Let X be a poset and let a be a maximal (resp. minimal) element of S. Define $S_a^\downarrow$ (resp. $S_a^\uparrow$) to be the disjoint union of the subsets $\{a\}$ and $S \setminus a$ with the smallest order relation which contains that on $S \setminus a$, and such that $a < N(a)$ (resp. $a > N(a)$). Or briefly, $S_a^\downarrow = (S \setminus a) \cup a$, where a is a minimal (resp. maximal) element of $S_a^\downarrow$ (resp. $S_a^\uparrow$) and $a < x$ in $S_a^\downarrow$ (resp. $a > x$ in $S_a^\uparrow$) iff $x \in N_S(a)$.

Proposition 1. The Tits forms of the posets S and $S_a^\downarrow$ (resp. $S_a^\uparrow$) are equivalent.

The proposition follows from the following (easily verifiable) equality for the Tits form $q(z)$ of S and the Tits form $q'(z)$ of $S_x^\downarrow$ (resp. $S_x^\uparrow$): $q(z) = q'(z')$, where $z'_0 = z_0 - z_a$, $z'_x = z_x$ for $x \neq a$ and $z'_a = -z_a$.

We call posets S and T (min, max)-*equivalent* if there are posets $S_1, \ldots, S_p$ with $p \geq 0$ such that, if one sets $S = S_0$ and $T = S_{p+1}$, then for each $i = 0, 1, \ldots, p$, either $S_{i+1} = (S_i)_x^{\downarrow}$ (with some maximal $x \in S_i$) or $S_{i+1} = (S_i)_y^{\uparrow}$ (with some minimal $y \in S_i$).

Directly from Proposition 1 it follows the following assertion.

Proposition 2. Let S and T be (min, max)-equivalent posets. Then their Tits forms are equivalent (and hence simultaneously positive or not).

Corollary 1. Let T be a poset and L be a chain. Then the Tits forms of the posets $T \coprod L$ and $[T < L]$ $(or [L < T])$ are simultaneously positive or not.

Indeed, if $L = \{a < b < \ldots < c\}$, then $(T \coprod L)_{ab\ldots c}^{\uparrow\uparrow\ldots\uparrow} = [T < L^{\mathrm{op}}] \cong [T < L]$ and $(T \coprod L)_{cb\ldots a}^{\downarrow\downarrow\ldots\downarrow} = [L^{\mathrm{op}} < T] \cong [L < T]$. And it remains to apply Proposition 2.

2.3. Statements on one-sided sums

A subset A of a poset S is said to be *upper* (resp. *lower*) if $x \in A$ whenever $x > y$ (resp. $x < y$) and $y \in A$. Recall that the posets T_1 and T_2 were defined in Subsection 2.1 (see Lemma 1).

Lemma 6. Let S be a poset of width 2 not containing a subset isomorphic to the poset T_1, and assume that S_0 is empty. Then S is a one-sided sum of two chains. If in addition S does not contain a subset isomorphic to the poset T_2, then the sum is semiminimax.

The assertions of the lemma are obvious if one takes into account that S has two maximal and two minimal elements and, as a poset of width 2, is a sum of two chains.

For any poset of width 2 we have the following assertion.

Lemma 7. Let S be a poset of width 2 not containing a subset isomorphic to the poset T_1. Then

a) S_0 is the union of an upper and a lower subsets;

b) S is a one-sided sum of two chains.

Proof. If *a*) did not hold, then there would be elements $c \in S_0$ and $a, b, d, e \in S \setminus S_0$ such that a and b (resp. d and e) are incomparable and $a < c, b < c, c < d, c < e$; so the subset $\{a, b, d, e\}$ would be isomorphic to T_1, a contradiction. Further, by the previous lemma $S \setminus S_0$ is a one-sided sum of two chains, say A and B with $B \ntriangleleft A$. Denote by S_{01} and S_{02} the lower and upper "parts" of S_0. Then S is a one-sided sum of the subsets $S_{01} \cup A = [S_{01} < A]$ and $B \cup S_{02} = [B < S_{02}]$ which are chains. $\square$

Obviously, if b) holds, then S does not contain a subset isomorphic to T_1; further, if the sum is semiminimax, then S also does not contain a subset isomorphic to T_2. So we have the following corollaries from two previous lemmas.

Corollary 2. A poset S of width 2 is a one-sided sum of two chains if and only if it contains no subset isomorphic to the poset T_1.

Corollary 3. The subset S° of a poset S of width 2 is a semiminimax one-sided sum of two chains if and only if S° (or equivalently, S) contains no subset isomorphic to the poset T_1 or T_2.

3. Proof of the sufficiency of Theorem 1

Let S be a poset of the form 1), 2) or 3). We prove that the Tits form of S is positive. By Corollary 1 the first case reduces to the second one (because, for chains X and Y, the poset $[X < Y]$ is a chain too, and any chain is a subset of some almost chain).

If S is an almost chain, then letting c and d to be its only pair of incomparable elements, one has the (easily verifiable) equality

$$2q_S(z) = z_0^2 + \sum_{\substack{x\neq c,d,\\ x\in S}} z_x^2 + (z_c - z_d)^2 + (z_0 - \sum_{x\in S} z_x)^2,$$

and so the form $q_S(z)$ is positive.

It remains to prove that the Tits form $q_S(z)$ is positive for S to be the right minimax sum of chains A and B. Let a and b be the minimal element of A and the maximal element of B, respectively. Then $S_b^{\downarrow}$ is the direct sum of the chain $L = B \setminus b$ and the almost chain $T = A \cup \{b\}$ with $b < A \setminus a$. Since $[L < T]$ is an almost chain, the Tits form of S is positive (via Proposition 1 and Corollary 1).

4. A complete list of the posets of order $n < 8$ with positive Tits form

In this section we give a complete list of the posets of order smaller than 8 (and width 2) with positive Tits form.

We denote by $\langle m\rangle_i$ the chain $\{1+i < 2+i < \cdots < m+i\}$ (for $m, i \geqslant 0$) and set $\langle m\rangle = \langle m\rangle_0$, $\langle m,n\rangle_i = \langle m\rangle_i \coprod \langle n\rangle_{m+i}$. Further, for nonempty chains L and L', we denote by $L \nearrow L'$ their right minimax sum of rank 1, and for posets X, Y and Z, we set $[X < Y < Z] = [[X < Y] < Z]$. Finally, we set $D_{ij} = \langle i,j\rangle_0$, $M_{ij} = \langle i\rangle \nearrow \langle j\rangle_i$ and $Q_{ij} = [\langle i\rangle < \langle 1,1\rangle_{i+1} < \langle j\rangle_{i+2}]$.

We have the following obvious assertion.

Proposition 3. Let S be a poset of width 2 and order smaller than 8 satisfying one of conditions 1)–3) of Theorem 1. Then S has positive Tits form (by the previous section) and is isomorphic to a poset D_{ij} with $i, j > 0$ and $i + j < 8$, or M_{ij} with $i, j > 0$ and $2 < i + j < 8$, or Q_{ij} with $i, j \geqslant 0$ and $i + j < 6$.

The following theorem give (together with Proposition 3) a complete list of posets of width 2 and order $n < 8$ with positive Tits form (by Lemma 5 it is sufficient to consider such posets up to 0-isomorphism and duality).

Theorem 2. Let S be a poset of width 2 and order $n < 8$. Then the Tits form of S is positive if and only if either one of conditions 1)–3) is satisfied, or S_0 is the union of an upper and a lower subsets and S is 0-isomorphic or 0-antiisomorphic to one of the following poset (which consists of the elements $1, \ldots, n$ and is a right sum of chains $\{1 \prec \ldots \prec i\}$ and $\{i+1 \prec \ldots \prec n\}$ with $i \leqslant n/2$) :

A) (of order 5)

$$
\begin{aligned}
R_1 &= \{2 \prec 3 \prec 4 \prec 5, 1 \prec 4\},\\
R_2 &= \{1 \prec 2 \prec 5, 3 \prec 4 \prec 5\},\\
R_3 &= \{1 \prec 2, 3 \prec 4 \prec 5, 1 \prec 4\},\\
R_4 &= \{1 \prec 2 \prec 5, 3 \prec 4 \prec 5, 1 \prec 4\};
\end{aligned}
$$

B) (of order 6)

$$
\begin{aligned}
R_5 &= \{2 \prec 3 \prec 4 \prec 5 \prec 6, 1 \prec 5\},\\
R_6 &= \{2 \prec 3 \prec 4 \prec 5 \prec 6, 1 \prec 4\},\\
R_7 &= \{1 \prec 2 \prec 6, 3 \prec 4 \prec 5 \prec 6\},\\
R_8 &= \{1 \prec 2, 3 \prec 4 \prec 5 \prec 6, 1 \prec 5\},\\
R_9 &= \{1 \prec 2, 3 \prec 4 \prec 5 \prec 6, 1 \prec 4\},\\
R_{10} &= \{1 \prec 2 \prec 6, 3 \prec 4 \prec 5 \prec 6, 1 \prec 5\},\\
R_{11} &= \{1 \prec 2 \prec 6, 3 \prec 4 \prec 5 \prec 6, 1 \prec 4\},\\
R_{12} &= \{1 \prec 2 \prec 5, 3 \prec 4 \prec 5 \prec 6, 1 \prec 4\},\\
R_{13} &= \{1 \prec 2 \prec 3, 4 \prec 5 \prec 6, 2 \prec 6\},\\
R_{14} &= \{1 \prec 2 \prec 3, 4 \prec 5 \prec 6, 1 \prec 5, 2 \prec 6\},
\end{aligned}
$$

C) (of order 7)

$$
\begin{aligned}
R_{15} &= \{2 \prec 3 \prec 4 \prec 5 \prec 6 \prec 7, 1 \prec 6\},\\
R_{16} &= \{2 \prec 3 \prec 4 \prec 5 \prec 6 \prec 7, 1 \prec 4\},\\
R_{17} &= \{1 \prec 2 \prec 7, 3 \prec 4 \prec 5 \prec 6 \prec 7\},\\
R_{18} &= \{1 \prec 2, 3 \prec 4 \prec 5 \prec 6 \prec 7, 1 \prec 6\},\\
R_{19} &= \{1 \prec 2, 3 \prec 4 \prec 5 \prec 6 \prec 7, 1 \prec 4\},\\
R_{20} &= \{1 \prec 2 \prec 7, 3 \prec 4 \prec 5 \prec 6 \prec 7, 1 \prec 6\},\\
R_{21} &= \{1 \prec 2 \prec 7, 3 \prec 4 \prec 5 \prec 6 \prec 7, 1 \prec 5\},\\
R_{22} &= \{1 \prec 2 \prec 7, 3 \prec 4 \prec 5 \prec 6 \prec 7, 1 \prec 4\},\\
R_{23} &= \{1 \prec 2 \prec 6, 3 \prec 4 \prec 5 \prec 6 \prec 7, 1 \prec 4\},\\
R_{24} &= \{1 \prec 2 \prec 5, 3 \prec 4 \prec 5 \prec 6 \prec 7, 1 \prec 4\},\\
R_{25} &= \{1 \prec 2 \prec 3, 4 \prec 5 \prec 6 \prec 7, 2 \prec 7\},\\
R_{26} &= \{1 \prec 2 \prec 3, 4 \prec 5 \prec 6 \prec 7, 1 \prec 6\},\\
R_{27} &= \{1 \prec 2 \prec 3, 4 \prec 5 \prec 6 \prec 7, 1 \prec 6, 2 \prec 7\},\\
R_{28} &= \{1 \prec 2 \prec 3, 4 \prec 5 \prec 6 \prec 7, 1 \prec 5, 2 \prec 7\},\\
R_{29} &= \{1 \prec 2 \prec 3 \prec 7, 4 \prec 5 \prec 6 \prec 7, 1 \prec 6\},\\
R_{30} &= \{1 \prec 2 \prec 3 \prec 7, 4 \prec 5 \prec 6 \prec 7, 1 \prec 5, 2 \prec 6\}.
\end{aligned}
$$

Proof. The necessity part follows from the Proposition 3 and the following one.

Proposition 4. Let S be a poset of width 2 and order $n < 8$ not containing a subset 0-isomorphic to the poset T_1, T_2 or T_3 (of Lemma 1). Then S is 0-isomorphic or 0-antiisomorphic to one of the following poset: D_{ij} with $j \geqslant i > 0$ and $i + j < 8$, M_{ij} with $j \geqslant i > 0$ and $2 < i + j < 8$, Q_{0j} with $0 \leqslant j < 6$, R_i with $i = 1, 2, \ldots, 30$, T_i for $i = 4, 5, 6, 7$ (see Proposition 3, Theorem 2 and Lemma 1).

Notice that each of the posets D_{ij}, M_{ij}, Q_{ij}, R_{ij} and T_i ($i \neq 1$) is a right sum of chains $\{1 \prec \ldots \prec i\}$ and $\{i+1 \prec \ldots \prec n\}$. And Proposition 4 can be easily proved by exhaustion, up to 0-isomorphism, of all right sums of two chains, having the order $n < 8$ (see Lemma 7). Such a proof can be done more simply if one first describes, up to isomorphism, the right sums that are semiminimax and without nodal elements (see Lemma 6), and after this, the others.

Now we prove the sufficiency part (of Theorem 2) showing that the posets R_1–R_{30} have positive Tits form. Because each poset R_j of order 5 or 6 is a subset of some poset R_i of order 7, it is sufficient to consider only the posets of order 7, i.e. $R_{15}, R_{16}, \ldots, R_{30}$.

We have the following equalities: $(R_{15})_2^{\uparrow} \cong R_{18}$, $(R_{18})_3^{\uparrow} \cong R_{26}$, $(R_{26})_4^{\uparrow} \cong R_{25}^{\mathrm{op}}$, $(R_{25})_3^{\downarrow} \cong R_{17}$, $(R_{17})_1^{\uparrow} \cong R_{19}^{\mathrm{op}}$ and $(R_{19})_2^{\downarrow} \cong R_{16}$; $(R_{24})_3^{\uparrow} \cong_0 R_{22}$, $(R_{22})_3^{\uparrow} \cong_0 R_{20}$ and $(R_{20})_3^{\uparrow} \cong R_{27}$; $(R_{23})_3^{\uparrow} \cong_0 R_{21}$, $(R_{21})_3^{\uparrow} \cong R_{28}$ and $(R_{28})_4^{\uparrow} \cong_0 R_{29}$. So each of the sets $\mathcal{S}_1 = \{R_i \,|\, i = 15, 16, 17, 18, 19, 25, 26\}$, $\mathcal{S}_2 = \{R_i \,|\, i = 20, 22, 24, 27\}$ and $\mathcal{S}_3 = \{R_i \,|\, i = 21, 23, 28, 29\}$ consists of pairwise 0-isomorphic or 0-antiisomorphic posets and hence it is sufficient to make sure (in view of Proposition 1 and Lemma 5) that R_{30} and, for instance, R_{15}, R_{23} and R_{24} have positive Tits form. To do this, we use the well known fact that the Tits form

$$g(t) = g(t_1, t_2, \dots, t_8) = \sum_{i=1}^{8} t_i^2 - \sum_{i=1}^{6} t_i t_{i+1} - t_3 t_8$$

of the Dynkin graph E_8, with the vertices $1, 2, \dots, 8$ and the edges $(1,2), (2,3), \dots, (6,7)$ and $(3,8)$, is positive (see [1]). For the Tits forms $\overline{q}_{15}(z), \overline{q}_{23}(z), \overline{q}_{24}(z)$ and $\overline{q}_{30}(z)$ of the posets R_{15}, R_{23}, R_{24} and R_{30}, where $z = (z_0, z_1, \dots, z_7)$, one has the following equalities:

$\overline{q}_{15}(z) = g(t)$, where $t_1 = -z_6, t_2 = -z_6 - z_7, t_3 = z_0 - z_6 - z_7, t_4 = z_2 + z_3 + z_4 + z_5, t_5 = z_3 + z_4 + z_5, t_6 = z_4 + z_5, t_7 = z_5, t_8 = z_1$;

$\overline{q}_{23}(z) = g(t)$, where $t_1 = z_2, t_2 = z_0 - z_1 - z_6 - z_7, t_3 = z_0 + z_4 + z_5 - z_6 - z_7, t_4 = z_0 + z_4 - z_6 - z_7, t_5 = z_0 - z_6 - z_7, t_6 = -z_6 - z_7, t_7 = -z_6, t_8 = z_3 + z_4 + z_5$;

$\overline{q}_{24}(z) = g(t)$, where $t_1 = z_2, t_2 = z_0 - z_1 - z_5 - z_6 - z_7, t_3 = z_0 + z_4 - z_5 - z_6 - z_7, t_4 = z_0 - z_5 - z_6 - z_7, t_5 = -z_5 - z_6 - z_7, t_6 = -z_5 - z_6, t_7 = -z_6, t_8 = z_3 + z_4$;

$\overline{q}_{30}(z) = g(t)$, where $t_1 = z_1 + z_2 + z_3, t_2 = z_0 + z_1 + z_3 - z_6 - z_7, t_3 = 2z_0 + z_1 - z_6 - 2z_7, t_4 = z_0 + z_1 - z_6 - 2z_7, t_5 = z_0 + z_1 - z_6 - z_7, t_6 = z_1 + z_4 + z_5, t_7 = z_1 + z_5, t_8 = z_0 - z_6 - z_7$.

Hence each of the Tits forms (of the four fix posets) is equivalent to the form $g(t)$ and thus is positive, as claimed. $\square$

5. Proof of the necessity of Theorem 1 for posets of order 8

Let S be a poset of width 2 and order 8 with positive Tits form. We prove that then for S one of condition 1)–3) holds. This assertion follows immediate from Lemmas 1–3 and the next proposition.

Proposition 5. Let S be a poset of width 2 and order 8 not containing subsets 0-isomorphic or 0-antiisomorphic to the posets T_1, T_2, ..., T_{20}. Then S satisfies condition $1), 2)$ or 3) (of Theorem 1).

Proof of Proposition. The proof depends on the number $s = |S_0|$ (the smaller s, the more complicated the proof). It is easy to see (taking into account Lemma 4 and the fact that a poset 0-isomorphic to some almost chain is an almost chain too) that the subset S_0 can be assumed to be upper. Note that $w(S) = 2$ implies $s \neq 7, 8$.

We first prove that if $s = 6, 5, 4, 3, 2$, then S satisfies condition 2), i.e. is an almost chain.

By Proposition 4 there are, up to isomorphism and duality, the following possibilities for the poset S° of order $8-s$ [2]:

in the case $s=6$, $S_1^\circ = D_{11}$; in the case $s=5$, $S_1^\circ = D_{12}$; in the case $s=4$, $S_1^\circ = D_{13}$, $S_2^\circ = D_{22}$, $S_3^\circ = M_{22}$; in the case $s=3$, $S_1^\circ = D_{14}$, $S_2^\circ = D_{23}$, $S_3^\circ = M_{23}$, $S_4^\circ = R_3$; in the case $s=2$, $S_1^\circ = D_{15}$, $S_2^\circ = D_{24}$, $S_3^\circ = D_{33}$, $S_4^\circ = M_{24}$, $S_5^\circ = M_{33}$, $S_6^\circ = R_8$, $S_7^\circ = R_9$, $S_8^\circ = R_{13}$, $S_9^\circ = R_{14}$.

In the case $s=6$, $S^\circ \cong S_1^\circ$ and therefore S is an almost chain. The cases $s=5,4,3,2$ are impossible since, for each s, every poset $S_{si} = [S_i^\circ < L_s]$, with S_i° running through all the posets listed in the corresponding case and $L_s = \{8-s+1 \prec 8-s+2 \prec \ldots \prec 8\}$ being a chain of order s, contains a, proper or not, subset isomorphic to some T_j: for $(s,i) = (4,2)$, $(3,2), (3,3),\ (2,2), (2,3), (2,4), (2,5),\ (2,6), (2,8),\ (2,9)$, S_{si} contains a subset $T \cong T_3$; for $(s,i) = (4,1), (3,1), (3,4)$, S_{si} contains a subset $T \cong T_4$; for $(s,i) = (5,1),\ (4,3),\ (2,1)$, $S_{si} \cong T_9, T_{13}, T_8$, respectively; for $(s,i) = (2,7)$, S_{si} contains a subset $T \cong_0 T_4$ ($T = S_{si} \setminus \{3\}$).

Now we prove that if $s=1$, then S satisfies condition 3) with one of the chains being one-element.

By Propositions 4 we have in this case the following possibilities for the subset S° of order 7 [3]: $S_1^\circ = D_{16}$, $S_2^\circ = D_{25}$, $S_3^\circ = D_{34}$, $S_4^\circ = M_{25}$, $S_5^\circ = M_{34}$, $S_6^\circ = R_{18}$, $S_7^\circ = R_{19}$, $S_8^\circ = R_{25}$, $S_9^\circ = R_{26}$, $S_{10}^\circ = R_{27}$, $S_{11}^\circ = R_{28}$.

In the case $S^\circ = S_1^\circ$, the poset S satisfies condition 3) with one of the chains being one-element. All the other cases are impossible since every poset $S_i = [S_i^\circ < L_1]$, where $L_1 = \{8\}$ is a one-element set and $i = 2,3,\ldots,14$, contains a, proper or not, subset 0-isomorphic to some $P = T_j$ ($j = 1,2,\ldots,20$): for $i = 3,5,8$, S_i contains a subset $T \cong T_7$; for $i=6$, S_i contains a subset $T \cong T_4$; for $i = 2,4,7$, $S_i \cong T_{10}, T_{12}, T_{11}$, respectively; for $i = 9,10,11$, S_i contains a subset $T \cong_0 T_3$ ($T = S_i \setminus \{4,5\}, S_i \setminus \{3,6\}, S_i \setminus \{4,7\}$, respectively).

Finally we prove that if $s=0$ then S satisfies condition 1) or 3).

We need the following statement which is an analog of Proposition 4 on posets of order 8 without nodal elements.

Proposition 6. *Let S be a poset of width 2 and order 8 not containing a subset isomorphic to the poset T_1 or T_2 (of Lemma 1), and let S has no nodal elements.*

Then S is isomorphic or antiisomorphic to one of the following poset:

[2]Since the posets T_i ($i = 1,2,\ldots,20$) are considered up to 0-isomorphism or 0-antiisomorphism (see the formulation of the proposition), S° can be described up to isomorphism and duality. Of course, we do not consider those S° that contain a subset 0-isomorphic or 0-antiisomorphic to some T_i.

[3]See footnote 2.

$S_1 = D_{17}$, $S_2 = D_{26}$, $S_3 = D_{35}$, $S_4 = D_{44}$,
$S_5 = M_{26}$, $S_6 = M_{35}$, $S_7 = M_{44}$,
$S_8 = \{1 \prec 2, 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8, 1 \prec 6\}$,
$S_9 = \{1 \prec 2, 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8, 1 \prec 5\}$,
$S_{10} = \{1 \prec 2 \prec 3, 4 \prec 5 \prec 6 \prec 7 \prec 8, 1 \prec 6\}$,
$S_{11} = \{1 \prec 2 \prec 3, 4 \prec 5 \prec 6 \prec 7 \prec 8, 1 \prec 5\}$,
$S_{12} = \{1 \prec 2 \prec 3, 4 \prec 5 \prec 6 \prec 7 \prec 8, 1 \prec 6, 2 \prec 8\}$,
$S_{13} = \{1 \prec 2 \prec 3, 4 \prec 5 \prec 6 \prec 7 \prec 8, 1 \prec 5, 2 \prec 8\}$,
$S_{14} = \{1 \prec 2 \prec 3 \prec 4, 5 \prec 6 \prec 7 \prec 8, 3 \prec 8\}$,
$S_{15} = \{1 \prec 2 \prec 3 \prec 4, 5 \prec 6 \prec 7 \prec 8, 1 \prec 6, 2 \prec 8\}$,
$S_{16} = \{1 \prec 2 \prec 3 \prec 4, 5 \prec 6 \prec 7 \prec 8, 1 \prec 6, 3 \prec 8\}$,
$S_{17} = T_{14}$, $S_{18} = T_{15}$, $S_{19} = T_{16}$, $S_{20} = T_{17}$,
$S_{21} = T_{18}$, $S_{22} = T_{19}$, $S_{23} = T_{20}$.

The proposition can be easily proved by exhaustion, up to isomorphism, of all semiminimax right sums (without nodal elements) of two chains, having order 8 (see Lemma 6).

Thus, we must consider the cases when $S = S_i$, $i = 1, 2, \dots, 16$.

The poset S satisfies condition 1) in the cases $S = S_1, S_2, S_3, S_4$ and condition 3) in the cases $S = S_5, S_6, S_7$. All the other cases are impossible since every poset S_i, $i = 8, 9, \dots, 16$, contains a (proper) subset 0-isomorphic (in fact, isomorphic if $i \neq 16$) to a poset $T = T_{p(i)}$ from Lemma 1. Namely, one has to take $p(i) = 4$ if $i = 8$, $p(i) = 5$ if $i = 9, 10, 12$, $p(i) = 6$ if $i = 11, 13, 15$, $p(i) = 7$ if $i = 14$ and $p(i) = 3$ if $i = 16$ (in the last case $T = S \setminus \{4, 5\}$).

□

6. Proof of the necessity of Theorem 1 for posets of order greater than 8

Through this section S is (as in the previous one) a poset of width 2.

We say that a poset S contains (no) subsets of the form T, where T is a fixed poset (with elements being natural numbers), if there is (no) subsets of S isomorphic to T.

We need the following assertion.

Proposition 7. A poset S of width 2 (and any order) satisfies one of condition 1)–3) of Theorem 1 if and only if it does not contain, up to duality, a subset one of the following forms:

a) $\{1 \prec 3, 1 \prec 4, 2 \prec 3, 2 \prec 4\}$;
b) $\{1 \prec 4, 2 \prec 3 \prec 4 \prec 5\}$;
c) $\{1 \prec 2 \prec 5, 3 \prec 4 \prec 5\}$;
d) $\{1 \prec 2, 1 \prec 4, 3 \prec 4 \prec 5\}$;
e) $\{1 \prec 2 \prec 5, 3 \prec 4 \prec 5, 1 \prec 4\}$;
f) $\{1 \prec 2 \prec 5, 3 \prec 4 \prec 5, 1 \prec 3\}$.

Proof. The necessity part is obvious, and we proceed to a proof of the sufficiency one. By $b^{\mathrm{op}})$–$f^{\mathrm{op}})$ we denote the posets dual to $b)$–$f)$.

Let S contains no subsets of the form $a)$–$f)$ and $b^{\mathrm{op}})$–$f^{\mathrm{op}})$, and let S is not decomposable (otherwise it satisfies condition 1)). Then S is a one-sided sum of two chains (of nonzero rank), say $A = \{a_1 < \cdots < a_p\}$ and $B = \{b_1 < \cdots < b_q\}$, where $p + q > 2\,(p, q > 0)$ (see Lemma 7). Without loss of generality one can assume that S is a right sum of them and $p \leqslant q$ (taking into account that one can replace S by S^{op} and renames the summands). The rank $r_0 = r_0(A, B)$ of S can not be greater than two, because otherwise we would have short pairs $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in A \times B$ with $x_1 < x_2 < x_3, y_1 < y_2 < y_3$ and this would give the subset $\{x_2, x_3, y_1, y_2, y_3\}$ of the form $e)$.

We consider the cases $r_0 = 1$ and $r_0 = 2$ separately.

The case $r_0 = 1$. Let (a_i, b_j) be the only short pair. If $p = 1, j = 2$ (resp. $p = 1, j = q$), then S satisfies condition 2) (resp. 3)). Otherwise (in the case $p = 1$), the poset S contains a subset of the type $b)$, namely, the subset $\{a_1, b_1, b_2, b_j, b_q\}$, a contradiction.

If either $p = 2$ and $i = 1, j = 1, q$, or $p = q = 2$ and $i = j = 2$, then S satisfies condition 3). Otherwise (in the case $p = 2$), the poset S contains a subset of the type $b), c)$ or $d)$. Indeed, if $i = 1$ and $j \neq 1, q$, then S contains the subset $\{a_1, a_2, b_1, b_j, b_q\}$ of the form $d)$; and if $i = 2$, $q > 2$ and $j = 2$ (resp. $j > 2$), then S contains the subset $\{a_1, a_2, b_1, b_2, b_3\}$ (resp. $\{a_1, a_2, b_1, b_2, b_j\}$) of the form $b)$ (resp. $c)$). Again a contradiction.

Finally, if $p > 2$ and $i = 91, j = q$, then S satisfies condition 3). Otherwise (in the case $p > 2$), the poset S contains a subset of the type $c), c^{\mathrm{op}}, d), d^{\mathrm{op}})$ or $b^{\mathrm{op}})$: if $i = p, j = q$ (resp. $i = 1, j = 1$), then S contains the subset $\{a_1, a_2, b_1, b_2, b_q\}$ (resp. $\{a_1, a_2, a_3, b_1, b_2\}$) of the form $c)$ (resp. $c^{\mathrm{op}})$); if $i \neq 1, p$ and $j = 1$ (resp. $j = q$), then S contains the subset $\{a_1, a_i, a_p, b_1, b_2\}$ (resp. $\{a_1, a_i, a_p, b_1, b_q\}$) of the form $b^{\mathrm{op}})$ (resp. $d^{\mathrm{op}})$); if $i \neq p$ and $j \neq 1, q$, then S contains the subset $\{a_i, a_p, b_1, b_j, b_q\}$ of the form $d)$; if $i = p$ and $j \neq 1, q$, then S contains the subset $\{a_1, a_p, b_1, b_j, b_q\}$ of the form $b)$. Again a contradiction.

The case $r_0 = 2$. Let (a_i, b_j) and (a_s, b_t), where $i < s, j < t$, be the corresponding short pairs. Then $s - i = 1$ and $t - j = 1$, because otherwise S would contain a subset of type $f^{\mathrm{op}})$ or $f)$ (that would give a contradiction). If $j = 1$, $s = p$, then S satisfies condition 2). If $j \neq 1$ (resp. $s \neq p$), then S contains the subset $\{a_i, a_s, b_1, b_j, b_t\}$ (resp. $\{a_i, a_s, a_p, b_j, b_t\}$) of the form e) (resp. e^{op})), a contradiction. □

Now it is easy to prove the following proposition.

Proposition 8. Let S be a poset of width 2 and order greater than 8 not containing subsets 0-isomorphic or 0-antiisomorphic to the posets T_1, T_2, …, T_{20}. Then S satisfies condition $1), 2)$ or 3) (of Theorem 1).

Indeed, let S be a poset of order at least 9 and width 2 with no subset 0-isomorphic or 0-antiisomorphic to T_1–T_{20}. If none of conditions 1)–3) were hold, then (by the sufficiency of Proposition 7) the poset S would contain a subset X of the form a)–f), and, for any subset of S of order 8 containing X^4, none of conditions 1)–3) would hold (by necessity of Proposition 7), a contradiction (to the necessity of Proposition 5).

The necessity of Theorem 1 for posets of order greater than 8 follows immediately from Proposition 8 and Lemmas 1–3.

7. Critical posets with respect to positivity of the Tits form

The following theorem give a complete list of critical posets of width 2 with respect to positivity of the Tits form, i. e. minimal posets of width 2 with nonpositive Tits form.

Theorem 3. A poset T of width 2 is critical with respect to positivity of the Tits form if and only if it is 0-isomorphic or 0-antiisomorphic to one of the posets T_i ($i = 1, 2, \ldots, 20$).

Indeed, by Lemmas 1–3, Propositions 5 and the sufficiency of Theorem 1, each poset 0-isomorphic or 0-antiisomorphic to T_i, $i \in \{1, 2, \ldots, 20\}$, is critical. The fact that there are no other critical posets follows from Propositions 4, 5 and 8.

References

[1] Gabriel P. *Unzerlegbare Darstellungen* // Manuscripts Math. – 1972. – V. 6. – P. 71–103,309.

[2] Brenner S. *Quivers with commutativity conditions and some phenomenology of forms* // Proc. of Intern. Conference of Representations of Algebras. – Carleton Univ., Ottawa, Ontario, 1974. – Paper N5.

[4] Such a subset exists because the order of each of the poset a)–f) (hence X) is smaller than 9.

[3] Drozd Yu. A. *Coxeter transformations and representations of partially ordered sets* // Funkc. Anal. i Priložen. – 1974. – V. 8. – P. 34–42 (in Russian).

[4] Kleiner M. M, Roiter A. V. *Representations of DGC* // Matrix problems / Kiev: Inst. Math. Acad. Sci. Ukrain. SSR, Kiev, 1977. – P. 5-70 (in Russian).

[5] Drozd Yu. A. *On tame and wild matrix problems* // Matrix problems / Kiev: Inst. Math. Acad. Sci. Ukrain. SSR, Kiev, 1977. – P. 104-114 (in Russian).

[6] *Matrix problems.* A collection of scientific works. Kiev: Inst. Math. Acad. Sci. Ukrain. SSR, Kiev, 1977. 166p. (in Russian).

[7] *Representations and quadratic forms.* A collection of scientific works. Kiev: Inst. Math. Acad. Sci. Ukrain. SSR, Kiev, 1979. 152p. (in Russian).

[8] Ringel C. M. *Tame algebras and integral quadratic forms.* Lecture notes in math. – 1099. 1984, 376p.

[9] Simson D. *Linear Representations of Partially Ordered Sets and Vector Space Category.* Gordon and Breach Science Publishers. 1992. 499p.

[10] Bondarenko V. M., Zavadskij A. G., Nazarova L. A. *On representations of tame partially ordered sets* // Representations and Quadratic Forms: Kiev: Inst. Math. Acad. Sci. Ukrain. SSR, 1979. – P. 75–106 (in Russian).

[11] Zavadskij A. G. *Series of representations and the Tits form* // Preprint N 27, 1981. – Kiev: Inst. Math. Acad. Sci. Ukrain. SSR. – P. 3–20 (in Russian).

[12] Bongartz K. *Algebras and quadratic forms* // J. London Math. Soc. (2). – 1983. – V. 28, N3. – P. 461–469.

[13] Kraft H., Reidtmann Ch. *Geometry of representations of quivers* // London Math. Soc. Lecture Note Ser. – 1986. – V. 116. – P. 109–145.

[14] Marmaridis N. *One point extensions of trees and quadratics forms* // Fund. Math. – 1990. – V. 134, N1. – P. 15–35.

[15] Kasjan S., Simson D. *Tame prinjective type and Tits form of two-peak posets.* I // J. Pure Appl. Algebra. – 1996. – V. 106, N 3. – P. 307–330.

[16] Kasjan S., Simson D. *Tame prinjective type and Tits form of two-peak posets.* II // J. Algebra. – 1997. – V. 187, N1. – P. 71–96.

[17] Drozd Yu. A. *Representations of bisected posets and reflection functors* // Algebras and modules, II (Geiranger, 1996), CMS Conf. Proc., 1998. – V. 24. – P. 153–165.

[18] Dräxler P., de la Peña J. A. *Tree algebras with non-negative Tits form* // Comm. Algebra. – 2000. – V. 8. – P. 3993–4012.

[19] Deng B. *Quasi-hereditary algebras and quadratic forms* // J. Algebra. – 2001. – V. 239, N2. – P. 438–459.

[20] Bondarenko V. M., Polishchuk A. M. *Minimax sums of posets and the quadratic Tits form* // Algebra Discrete Math. – 2004. – N 1 – P. 17–36.

[21] Bondarenko V. M. *On* (min,max)*-equivalence of posets and applications to the Tits forms* // Bull. of the University of Kiev (series: Physics and Mathematics). – 2005. – N 1 – P. 24–25.

Contact information

V. M. Bondarenko Institute of Mathematics, National Academy of Sciences of Ukraine, Tereshchenkivska 3, 01601 Kyiv, Ukraine
E-Mail: vit-bond@imath.kiev.ua

M. V. Styopochkina Kyiv Taras Shevchenko University, Volodymyrs'ka 64, Kyiv, 01033, Ukraine
E-Mail: StMar@ukr.net