On positive (non-negative) Tits form and (min, max)-equivalence of posets

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The quadratic Tits form, introduced by P. Gabriel for quivers, Yu. A. Drozd for posets, and A. V. Roiter for a wide class of matrix problems, plays an important role in representation theory. The report is devoted to study the structure of posets with positive (non-negative) Tits form; here we consider only finite posets.

Let S be a poset. For $x \in S$, we denote by $N(x)$ the set of all elements of S incomparable to x; for subsets A and B of S, we write $A < B$ if $a < b$ for any $a \in A$ and $b \in B$. Further, for a minimal (resp. maximal) element x of S, we denote by $S_x^\uparrow$ (resp. $S_x^\downarrow$) the disjoint union of the subsets $\{x\}$ and $S \setminus \{x\}$ with the smallest order relation which contains that given on $S \setminus \{x\}$, and such that $x > N(x)$ (resp. $x < N(x)$). Posets S and T is called (min, max)-*equivalent*, if there are posets $X_1, \dots, X_p$ with $p \geq 0$ such that, if one sets $S = X_0$ and $T = X_{p+1}$, then for each $i = 0, 1, \dots, p$, either $X_{i+1} = (X_i)_x^\downarrow$, with some maximal $x \in X_i$, or $X_{i+1} = (X_i)_y^\uparrow$, with some minimal $y \in X_i$ (see [1]); in this case we write $S \cong_{(\min,\max)} T$.

Let S be a poset (without the element 0) and let $\mathbb{Z}^{S\cup 0} = \{z = (z_i) \mid z_i \in \mathbb{Z}, i \in S \cup 0\}$, where $\mathbb{Z}$ denotes the integer numbers. The *Tits quadratic form* of S is by definition the form $q_S : \mathbb{Z}^{S\cup 0} \to \mathbb{Z}$ defined by the equality

$$q_S(z) = z_0^2 + \sum_{i\in S} z_i^2 + \sum_{i<j, i,j\in S} z_i z_j - z_0 \sum_{i\in S} z_i.$$

In [1] it is proved that the Tits forms of (min, max)-equivalent posets are equivalent.

The Tits form, as an arbitrary one with coefficient in $\mathbb{Z}$, is called *weakly positive* (resp. *weakly nonnegative*) if it takes a positive (resp. nonnegative) value on every nonzero vector $z \in \mathbb{Z}^{S\cup 0}$ with nonnegative coordinates.

A poset S is said to be *P-critical* (resp. *WP-critical*) if the Tits form of any proper subposet of S is positive (resp. weakly positive), but that of S is not.

Theorem 1. *A poset S is P-critical if and only if it is* (min, max)*-equivalent to a WP-critical poset.*

Theorem 2. *Let S be a poset. The following conditions are equivalent:*

(1) *The Tits form of S is positive.*

(2) *The Tits form of any poset $T \cong_{(\min,\max)} S$ is weakly positive.*

Similar results hold for posets with nonnegative and weakly nonnegative Tits forms.

References

[1] V. M. Bondarenko, *On* (min, max)-*equivalence of posets and applications to the Tits forms*, Bulletin of the University of Kiev (Series: Physics & Mathematics), 2005, N1, P. 11–13.

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