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SOLUTION OF NORMALLY SOLVABLE OPERATOR EQUATIONS IN A HILBERT SPACE

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We find a formula for the unique pseudoinverse of a normally solvable operator, establish conditions for the existence of a unique solution of a normally solvable equation, and obtain its representation in a Hilbert space. We also introduce the notion of one-sided pseudoinverse operators for normally solvable operators acting in Hilbert spaces and consider methods for their construction.

In the theory of ordinary differential equations and functional differential equations, many problems can be reduced to an operator equation $Lx = y$ with a linear, bounded, normally solvable operator L. This representation allows one to set aside specific difficulties typical of every individual problem and to analyze these problems using methods of the theory of operators and functional analysis, focusing on the investigation of their general properties. In the course of this analysis, there arises the problem of the construction of generalized inverse and pseudoinverse operators for normally solvable operators in Banach and Hilbert spaces.

Preliminary Information

Assume that a linear, bounded, normally solvable operator L acts from a real Hilbert space $\mathbf{H}_1$ into a real Hilbert space $\mathbf{H}_2$, $L\colon \mathbf{H}_1 \to \mathbf{H}_2$. According to [1, 2], an operator $L^-\colon \mathbf{H}_2 \to \mathbf{H}_1$ that possesses the properties

$$\begin{aligned} LL^-L &= L, \\ L^-LL^- &= L^- \end{aligned} \tag{1}$$

is called a generalized inverse of the operator L.

This operator is not uniquely defined. However, by using geometric properties of Hilbert spaces (the presence of a scalar product in these spaces and, as a consequence, their unique decomposability into direct sums of orthogonal subspaces, and the isomorphism of dual spaces), one can obtain subtler results on the generalized inversion of normally solvable operators in Hilbert spaces, namely, one can choose a unique pseudoinverse operator from the set of generalized inverses L^- of the operator L [3–5].

An operator $L^+\colon \mathbf{H}_2 \to \mathbf{H}_1$ that possesses the properties [3, 4]

$$\begin{aligned} LL^+L &= L, \\ L^+LL^+ &= L^+, \\ (LL^+)^* &= LL^+ = I_{\mathbf{H}_2} - P_{N(L^*)}, \\ (L^+L)^* &= L^+L = I_{\mathbf{H}_1} - P_{N(L)} \end{aligned} \tag{2}$$

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is called pseudoinverse to the operator L in the Moore–Penrose sense. Here, $P_{N(L)}: \mathbf{H}_1 \to N(L)$ and $P_{N(L^*)}: \mathbf{H}_2 \to N(L^*)$ are the orthoprojectors of an operator $L: \mathbf{H}_1 \to \mathbf{H}_2$ and its adjoint $L^*: \mathbf{H}_2^* \to \mathbf{H}_1^*$ to the null spaces $N(L)$ and $N(L^*)$, respectively.

It is known [6] that a space dual to a Hilbert space coincides with it up to isomorphism, i.e., $\mathbf{H}_1^* = \mathbf{H}_1$ and $\mathbf{H}_2^* = \mathbf{H}_2$. It was shown in [1, p. 139] that, since the null space $N(L) \subset \mathbf{H}_1$ and the image $R(L) \subset \mathbf{H}_2$ of the operator L are closed and any closed subset of a Hilbert space is complementable, L is a generalized invertible operator and there exist orthoprojectors $P_{N(L)}: \mathbf{H}_1 \to N(L)$, $LP_{N(L)} = 0$, and $P_{N(L^*)}: \mathbf{H}_2 \to N(L^*)$, $L^* P_{N(L^*)} = 0$, that induce a decomposition of $\mathbf{H}_1$ and $\mathbf{H}_2$ into direct orthogonal sums [7], namely,

$$\begin{aligned} \mathbf{H}_1 &= N(L) \oplus R(L^*), \\ \mathbf{H}_2 &= N(L^*) \oplus R(L), \end{aligned} \tag{3}$$

where $N(L) = P_{N(L)}\mathbf{H}_1$, $N(L^*) = P_{N(L^*)}\mathbf{H}_2$, $R(L) = (I_{\mathbf{H}_2} - P_{N(L)})\mathbf{H}_2$, and $R(L^*) = (I_{\mathbf{H}_1} - P_{N(L)})\mathbf{H}_1$.

Statement of the Problem

Consider the problem of finding conditions for the existence of solutions of the equation

$$Lx = y \tag{4}$$

and the construction of these solutions; here, $L: \mathbf{H}_1 \to \mathbf{H}_2$ is a linear, bounded, normally solvable operator.

In the present paper, we pose the following problems: Construct one-sided pseudoinverse operators L_r^+ and L_l^+ and, on their basis, a unique pseudoinverse operator L^+. Using orthoprojectors and the pseudoinverse operator L^+, find a criterion for the solvability of equations with a linear, bounded, normally solvable operator L and obtain expressions for these solutions.

Intermediate Result

Under the conditions of the problem posed, three cases are possible for the null spaces $N(L)$ and $N(L^*)$.

Case 1. The subspace $N(L)$ is linearly isomorphic to the subspace $N_1(L^*) \subset N(L^*)$, $N(L) \cong N_1(L^*)$. This implies that there exist

(i) a linear, bounded, invertible operator $J_1: N(L) \to N_1(L^*)$ such that $J_1 \times N(L) = N_1(L^*)$ and $J_1^{-1} N_1(L^*) = N(L)$;

(ii) an orthoprojector $P_{N_1(L^*)}: \mathbf{H}_2 \to \mathbf{H}_2$ that decomposes the subspace $N(L^*)$ into the direct sum of closed subspaces

$$N(L^*) = N_1(L^*) \oplus N_2(L^*), \tag{5}$$

where $N_1(L^*) = P_{N_1(L^*)}\mathbf{H}_2$, $N_2(L^*) = P_{N_2(L^*)}\mathbf{H}_2$, and $P_{N_2(L^*)} = P_{N(L^*)} - P_{N_1(L^*)}$ is an orthoprojector.

Case 2. The subspace $N_1(L) \subset N(L)$ is linearly isomorphic to the subspace $N(L^*)$, $N_1(L) \cong N(L^*)$. In this case, there exist

(i) a linear, bounded, invertible operator $J_2: N_1(L) \to N(L^*)$ such that $J_2 \times N_1(L) = N(L^*)$ and $J_2^{-1} N(L^*) = N_1(L)$;

(ii) an orthoprojector $P_{N_1(L)}: \mathbf{H}_1 \to \mathbf{H}_1$ that decomposes the subspace $N(L)$ into the direct sum of closed subspaces

$$N(L) = N_1(L) \oplus N_2(L), \tag{6}$$

where $N_1(L) = P_{N_1(L)}\mathbf{H}_1$, $N_2(L) = P_{N_2(L)}\mathbf{H}_1$, and $P_{N_2(L)} = P_{N(L)} - P_{N_1(L)}$ is an orthoprojector.

Case 3. The subspace $N(L)$ is linearly isomorphic to the subspace $N(L^*)$, $N(L) \cong N(L^*)$.

In this case, there exists a linear, bounded, invertible operator $J_3: N(L) \to N(L^*)$ such that $J_3 N(L) = N(L^*)$ and $J_3^{-1} N(L^*) = N(L)$.

Denote the extensions of the operators J_i, $i = 1, 2, 3$, to the space $\mathbf{H}_1$ by $\overline{P}_{N_1(L^*)}: \mathbf{H}_1 \to N_1(L^*) \subseteq N(L^*)$ and the extensions of the operators J_i^{-1}, $i = 1, 2, 3$, to the space $\mathbf{H}_2$ by $\overline{P}_{N_1(L)}: \mathbf{H}_2 \to N_1(L) \subseteq N(L)$. In Case 3, we have $N_1(L^*) \equiv N(L^*)$ and $N_1(L) \equiv N(L)$, and, therefore, $\overline{P}_{N_1(L^*)} \equiv \overline{P}_{N(L^*)}$ and $\overline{P}_{N_1(L)} \equiv \overline{P}_{N(L)}$.

Lemma 1. *On the subspace $\mathbf{H}_2 \ominus N_2(L^*)$, the operator $\overline{L} = L + \overline{P}_{N_1(L^*)}$ has the bounded inverse*

$$\overline{L}_{l,r}^{-1} = \begin{cases} (L + \overline{P}_{N_1(L^*)})_l^{-1} & \textit{(left) if} \quad N(L) \cong N_1(L^*) \subset N(L^*), \\ (L + \overline{P}_{N(L^*)})_r^{-1} & \textit{(right) if} \quad N(L) \supset N_1(L) \cong N(L^*). \end{cases}$$

The general form of the one-sided inverse operators $\overline{L}_{l_0,r_0}^{-1}$ is given by the formula

$$\overline{L}_{l_0,r_0}^{-1} = \begin{cases} \overline{L}_l^{-1}(I_{\mathbf{H}_2} - \widetilde{\mathcal{P}}_{N_2(L^*)}) & \textit{(left) if} \quad N(L) \cong N_1(L^*) \subset N(L^*), \\ (I_{\mathbf{H}_1} - \widetilde{\mathcal{P}}_{N_2(L)})\overline{L}_r^{-1} & \textit{(right) if} \quad N(L) \supset N_1(L) \cong N(L^*), \end{cases}$$

where $\widetilde{\mathcal{P}}_{N_2(L^*)}: \mathbf{H}_2 \to N_2(L^*)$ and $\widetilde{\mathcal{P}}_{N_2(L)}: \mathbf{H}_1 \to N_2(L)$ are arbitrary infinite-dimensional bounded projectors.

Proof. Let $N(L)$ be isomorphic to the subspace $N_1(L^*) \subset N(L^*)$.

We show that the operator $\overline{L} = L + \overline{P}_{N_1(L^*)}$ has a bounded left inverse. Since the subspaces $R(L)$ and $R(P_{N_1(L^*)}) = N_1(L^*)$ are closed, it follows from (3) and (5) that $R(\overline{L}) = R(L) \cup N_1(L^*)$ is closed. Since any closed subspace of a Hilbert space is complementable, for the existence of a left inverse $\overline{L}_l^{-1}$ it is necessary and sufficient that [1]

$$N(\overline{L}) = N(L + \overline{P}_{N_1(L^*)}) = \{0\}.$$

Assume that there exists $x_0 \in \mathbf{H}_1$, $x_0 \neq 0$, such that

$$(L + \overline{P}_{N_1(L^*)})x_0 = Lx_0 + \overline{P}_{N_1(L^*)}x_0 = 0. \tag{7}$$

Using (7), we obtain

$$Lx_0 \in R(L), \quad \overline{P}_{N_1(L^*)}x_0 \in N_1(L^*).$$

The subspaces $R(L)$ and $N(L^*)$ complement one another, $R(L)\bigcap N(L^*) = 0$, and $N_1(L^*) \subset N(L^*)$. Therefore, $R(L)\bigcap N_1(L^*) = \{0\}$, which implies that they have only one common element (the zero element), i.e., $Lx_0 = 0$ and $\overline{P}_{N_1(L^*)}x_0 = 0$. This implies that $x_0 \in N(L)$ and $x_0 \in N(\overline{P}_{N_1(L^*)}) \subset R(L^*)$. Since the subspaces $N(L)$ and $R(L^*)$ complement one another, using (3) we get $N(L) \cap R(L^*) = \{0\}$. This yields $x_0 = 0$. The contradiction obtained proves that $N(\overline{L}) = \{0\}$.

Thus, the operator $L + \overline{P}_{N_1(L^*)}$ has a left inverse.

Since the image $R(\overline{L}) = R(L) \oplus N_1(L^*)$ of the operator $\overline{L}$ does not coincide with the entire space $\mathbf{H}_2$, one cannot speak of the boundedness of the operator $\overline{L}_l^{-1}$ on the entire space $\mathbf{H}_2$. Since the subspace $\mathbf{H}_2 \ominus N_2(L^*)$ is closed, it is a space. The operator $\overline{L}$ establishes a one-to-one correspondence between the spaces $\mathbf{H}_1$ and $\mathbf{H}_2 \ominus N_2(L^*)$. Therefore, according to the Banach theorem [8], the boundedness of the operator $\overline{L}_l^{-1}$ is guaranteed only if we consider its action from the space $\mathbf{H}_2 \ominus N_2(L^*)$ onto the space $\mathbf{H}_1$.

The left inverse operator $\overline{L}_l^{-1}$ is not uniquely defined. Using the results of [1], we represent the collection of left inverse operators in the following general form:

$$\overline{L}_{l_0}^{-1} = \overline{L}_l^{-1}\mathcal{P}_{R(\overline{L})},$$

where $\mathcal{P}_{R(\overline{L})}$ is an arbitrary bounded projector to the image of the operator $\overline{L}$. It follows from (5) that the projector $I_{\mathbf{H}_2} - \widetilde{\mathcal{P}}_{N_2(L^*)}$ possesses this property, i.e., $R(I_{\mathbf{H}_2} - \widetilde{\mathcal{P}}_{N_2(L^*)}) = R(\overline{L})$, where $\widetilde{\mathcal{P}}_{N_2(L^*)}\colon \mathbf{H}_2 \to N_2(L^*)$ is an arbitrary infinite-dimensional bounded projector, which can be constructed in the general form by using the Sobczyk lemma [9]. This implies that the family of left inverse operators admits the representation

$$\overline{L}_{l_0}^{-1} = \overline{L}_l^{-1}(I_{\mathbf{H}_2} - \widetilde{\mathcal{P}}_{N_2(L^*)}).$$

Now let $N(L^*)$ be isomorphic to the subspace $N_1(L) \subset N(L)$. We show that the operator $\overline{L} = L + \overline{P}_{N(L^*)}$ has a bounded right inverse. Since $N(L)$ is complementable in $\mathbf{H}_1$, by virtue of (3) and (6) the subspace $N(\overline{L})$ is complementable in $\mathbf{H}_1$. Thus, for the proof of the existence of a right inverse operator, it is necessary and sufficient to show that [1]

$$R(\overline{L}) = R(L + \overline{P}_{N_1(L^*)}) \equiv \mathbf{H}_2.$$

Since $N(L^*)$ is isomorphic to $N_1(L) \subset N(L)$, we have $\overline{P}_{N_1(L^*)} \equiv \overline{P}_{N(L^*)}\colon \mathbf{H}_1 \to N(L^*)$. By the definition of the operators $\overline{L}$ and $\overline{P}_{N(L^*)}$, for an arbitrary element $x \in \mathbf{H}_1$ we have

$$\overline{L}x = Lx + \overline{P}_{N(L^*)}x,$$

where $Lx \in R(L)$ and $\overline{P}_{N(L^*)}x \in N(L^*)$. Since the subspaces $R(L)$ and $N(L^*)$ complement one another in the Hilbert space $\mathbf{H}_2$, we have $R(\overline{L}) \equiv \mathbf{H}_2$.

Thus, the operator $L + \overline{P}_{N_1(L^*)}$ has a right inverse.

Since the operator $\overline{L}$ establishes a one-to-one correspondence between the spaces $\mathbf{H}_1 \ominus N_2(L)$ and $\mathbf{H}_2$, it follows from the Banach theorem [8] that the right inverse operator $\overline{L}_r^{-1}$ is bounded.

The right inverse operator is also not uniquely defined. Using the results of [1], we represent the collection of right inverse operators in the following general form:

$$\overline{L}_{r_0}^{-1} = \mathcal{P}_{N(\overline{L})} \times \overline{L}_r^{-1},$$

where $\mathcal{P}_{N(\overline{L})}$ is an arbitrary projector that has the property $N(\mathcal{P}_{N(\overline{L})}) = N(\overline{L})$. It follows from (6) that the projector $I_{\mathbf{B}_1} - \widetilde{\mathcal{P}}_{N_2(L)}$ possesses this property, i.e., $N(I_{\mathbf{B}_1} - \widetilde{\mathcal{P}}_{N_2(L)}) = N(\overline{L})$, where $\widetilde{\mathcal{P}}_{N_2(L)}\colon \mathbf{H}_1 \to N_2(L)$

is an arbitrary infinite-dimensional bounded projector, which can be constructed in the general form by using the Sobczyk lemma [9]. This implies that the left inverse operators admit the following general representation:

$$\overline{L}_{r0}^{-1} = (I_{\mathbf{H}_1} - \widetilde{\mathcal{P}}_{N_2(L)})\overline{L}_r^{-1}.$$

The lemma is proved.

Remark 1. If L is a Noetherian operator $(\operatorname{ind} L = \dim\ker L - \dim\ker L^* = s - k < \infty)$, then Lemma 1 reduces to Lemma 2.4 in [10, p. 66].

In Case 3, where the subspace $N(L)$ is isomorphic to $N(L^*)$, the following statement is true:

Lemma 2. *The operator* $\overline{L} = L + \overline{P}_{N(L^*)}$ *has the bounded inverse*

$$\overline{L}^{-1} = (L + \overline{P}_{N(L^*)})^{-1}.$$

Proof. If $N(L)$ is isomorphic to $N(L^*)$, then $N_1(L) \equiv N(L)$ and $N_1(L^*) \equiv N(L^*)$. In this case, the operator $\overline{L}$ has left and right inverses, and, consequently, there exists a unique bounded inverse $\overline{L}^{-1}$.

The lemma is proved.

Remark 2. If a normally solvable operator L acts from a Hilbert space $\mathbf{H}$ into itself and $N(L)$ is isomorphic to $N(L^*)$, then it is called a reducible invertible operator. In this case, the lemma coincides with Theorem 1.6 in [11, p. 28].

Remark 3. If L is a Fredholm operator $(\operatorname{ind} L = 0)$, then Lemma 2 coincides with the known Schmidt lemma [8, p. 231].

Consider some relations for the "skew" projectors $\mathcal{P}_{N(L)}$ and $\mathcal{P}_{N(L^*)}$, orthoprojectors $P_{N(L)}$ and $P_{N(L^*)}$, linear operators $\overline{P}_{N_1(L)}$ and $\overline{P}_{N_1(L^*)}$, and operators $\overline{L}_{l_0,r_0}^{-1}$.

Lemma 3. *The orthoprojectors* $P_{N(L)}$ *and* $P_{N(L^*)}$ *and operators* $\overline{P}_{N_1(L)}$ *and* $\overline{P}_{N_1(L^*)}$ *satisfy the relations*

$$P_{N(L^*)}\overline{P}_{N_1(L^*)} = \overline{P}_{N_1(L^*)}P_{N(L)} = \overline{P}_{N_1(L^*)},$$

$$P_{N(L)}\overline{P}\,\overline{P}_{N_1(L)} = \overline{P}_{N_1(L)}P_{N(L^*)} = \overline{P}_{N_1(L)}. \tag{8}$$

Proof. We prove the first relation in (8). Let $x \in \mathbf{H}_1$. Then $\overline{P}_{N_1(L^*)}x \in N_1(L^*) \subset N(L^*)$ and, hence, $P_{N(L^*)}\overline{P}_{N_1(L^*)}x = \overline{P}_{N_1(L^*)}x$ because $P_{N(L^*)}N(L^*) = N(L^*)$. Therefore, $P_{N(L^*)}\overline{P}_{N_1(L^*)} = \overline{P}_{N_1(L^*)}$.

Let $x \in N(L)$. Then $P_{N(L)}x = x$. Acting by the operator $\overline{P}_{N_1(L^*)}$ on both sides of the last equality, we obtain $\overline{P}_{N_1(L^*)}P_{N(L)}x = \overline{P}_{N_1(L^*)}x$. Therefore, $\overline{P}_{N_1(L^*)}P_{N(L)} = \overline{P}_{N_1(L^*)}$.

The second relation in (8) is proved by analogy.

Note that Lemma 3 remains true if the orthoprojectors $P_{N(L)}$ and $P_{N(L^*)}$ are replaced by the "skew" projectors $\mathcal{P}_{N(L)}$ and $\mathcal{P}_{N(L^*)}$.

Lemma 4. *The orthoprojectors $P_{N(L)}$ and $P_{N(L^*)}$ and "skew" projectors $\mathcal{P}_{N(L)}$ and $\mathcal{P}_{N(L^*)}$ satisfy the relations*

$$P_{N(L)}\mathcal{P}_{N(L)} = \mathcal{P}_{N(L)}, \quad \mathcal{P}_{N(L)}P_{N(L)} = P_{N(L)},$$
$$\mathcal{P}_{N(L^*)}P_{N(L^*)} = \mathcal{P}_{N(L^*)}, \quad P_{N(L^*)}\mathcal{P}_{N(L^*)} = P_{N(L^*)}. \tag{9}$$

Proof. We prove the first relation in (9). Let $x_0 = \mathcal{P}_{N(L)}x$ for any $x \in \mathbf{H}_1$ and $P_{N(L)}x_0 = x_0$ for any $x_0 \in N(L)$. Then, replacing x_0 in the last equality by its value $\mathcal{P}_{N(L)}x$, we obtain

$$P_{N(L)}\mathcal{P}_{N(L)}x = \mathcal{P}_{N(L)}x \quad \forall x \in \mathbf{H}_1.$$

The second relation is proved by analogy.

It has been noted above that the spaces $\mathbf{H}_i^*$, $i = 1, 2$, coincide with the spaces $\mathbf{H}_i$, $i = 1, 2$, up to isomorphism. The dual spaces $\mathbf{H}_i^*$ are spaces of row vectors y^* and x^*. The projector $\mathcal{P}_{N(L^*)}$ $(L^*\mathcal{P}_{N(L^*)} = 0)$ acts on a row vector $y^* \in \mathbf{H}_2^*$ according to the rule $y^*\mathcal{P}_{N(L^*)}^*$. The orthoprojector $P_{N(L^*)}$, in view of its self-adjointness, acts on this row vector according to the rule $y^*P_{N(L^*)}$. Then a relation analogous to the first relation has the form

$$y^*P_{N(L^*)}\mathcal{P}_{N(L^*)}^* = y^*\mathcal{P}_{N(L^*)}^*.$$

Applying the operation of conjugation to both sides of the last equality, we obtain the third relation in (9).

The fourth relation is proved by analogy.

The lemma is proved.

Further, we establish some properties of the operators $\overline{L}_{l_0,r_0}^{-1}$ and L.

Lemma 5. *The operator $\overline{L}_{l_0,r_0}^{-1}$ satisfies the relations*

$$L\overline{L}_{l_0,r_0}^{-1} = I_{\mathbf{H}_2} - \mathcal{P}_{N(L^*)},$$
$$\overline{L}_{l_0,r_0}^{-1}L = I_{\mathbf{H}_1} - \mathcal{P}_{N(L)}, \tag{10}$$

where $I_{\mathbf{H}_1}$ and $I_{\mathbf{H}_2}$ are the identity operators in the spaces $\mathbf{H}_1$ and $\mathbf{H}_2$, respectively, and $\mathcal{P}_{N(L)}\colon \mathbf{H}_1 \to N(L)$ and $\mathcal{P}_{N(L^*)}\colon \mathbf{H}_2 \to N(L^*)$ are bounded projectors.

Proof. It follows from the definition of a right inverse operator $\overline{L}_{r_0}^{-1}$ that if it exists, then [1]

$$\overline{L}\,\overline{L}_{r_0}^{-1} = I_{\mathbf{H}_2},$$

$$\overline{L}_{r_0}^{-1}\overline{L} = I_{\mathbf{H}_2} - \mathcal{P}_{N_2(L)},$$

where $\mathcal{P}_{N_2(L)}$ is a bounded projector to the subspace $N_2(L) \subset N(L)$. If a left inverse operator $\overline{L}_{l_0}^{-1}$ exists, then

$$\overline{L}\,\overline{L}_{l_0}^{-1} = I_{\mathbf{H}_1} - \mathcal{P}_{N_2(L^*)},$$

$$\overline{L}_{l_0}^{-1}\,\overline{L} = I_{\mathbf{H}_1},$$

where $\mathcal{P}_{N_2(L^*)}$ is a bounded projector to the subspace $N_2(L^*) \subset N(L^*)$.

Since $\mathcal{P}_{N(L^*)}\,\overline{P}_{N_1(L^*)} = \overline{P}_{N_1(L^*)}$, $L\mathcal{P}_{N_2(L)} = 0$, and $\mathcal{P}_{N(L^*)}L = 0$, acting by the operator $\overline{L}$ on both sides of the first relation in (10) from the right, we obtain the identity

$$L = LI_{\mathbf{H}_1} = L\,\overline{L}_{l_0}^{-1}\,\overline{L} \equiv (I_{\mathbf{H}_2} - \mathcal{P}_{N(L^*)})\overline{L} = (I_{\mathbf{H}_2} - \mathcal{P}_{N(L^*)})(L + \overline{P}_{N_1(L^*)})$$

$$= L - \mathcal{P}_{N(L^*)}L + \overline{P}_{N_1(L^*)} - \mathcal{P}_{N(L^*)}\overline{P}_{N_1(L^*)} = L + \overline{P}_{N_1(L^*)} - \overline{P}_{N_1(L^*)} = L,$$

which proves the indicated relation.

Further, since $L\mathcal{P}_{N(L)} = 0$ and $\mathcal{P}_{N(L^*)}L = 0$, acting by the operator L on the second relation in (10) from the left, we obtain the identity

$$L = L + \mathcal{P}_{N(L^*)}L = (I_{\mathbf{H}_2} - \mathcal{P}_{N(L^*)})L = L\overline{L}_{l_0,r_0}^{-1}L \equiv L(I_{\mathbf{H}_1} - \mathcal{P}_{N(L)}) = L,$$

which proves the indicated relation.

The lemma is proved.

Using the lemmas proved above, we can propose a procedure for the construction of one-sided pseudoinverse operators for a normally solvable operator.

Left and Right Pseudoinverse Operators for a Normally Solvable Operator

Definition 1. *An operator $L_r^+ \colon \mathbf{H}_2 \to \mathbf{H}_1$ that satisfies the conditions*

$$LL_r^+L = L,$$

$$L_r^+LL_r^+ = L_r^+, \tag{11}$$

$$(LL_r^+)^* = LL_r^+ = I_{\mathbf{H}_2} - P_{N(L^*)}$$

is called a right pseudoinverse of a normally solvable operator L.

Definition 2. *An operator $L_l^+ \colon \mathbf{H}_2 \to \mathbf{H}_1$ that satisfies the conditions*

$$LL_l^+L = L,$$

$$L_l^+LL_l^+ = L_l^+, \tag{12}$$

$$(L_l^+L)^* = L_l^+L = I_{\mathbf{H}_1} - P_{N(L)}$$

is called a left pseudoinverse of a normally solvable operator L.

It is obvious that an operator that is simultaneously a left pseudoinverse and a right pseudoinverse is a pseudoinverse operator in the Moore–Penrose sense.

The theorems presented below describe the structure of one-sided pseudoinverse operators.

Theorem 1. *The operator*

$$L_r^+ = \overline{L}_{l_0,r_0}^{-1}(I_{\mathbf{H}_2} - P_{N(L^*)}) = \begin{cases} \overline{L}_{l_0}^{-1}(I_{\mathbf{H}_2} - P_{N(L^*)}) & \text{if} \quad N(L) \cong N_1(L^*) \subset N(L^*), \\ \overline{L}_{r_0}^{-1}(I_{\mathbf{H}_2} - P_{N(L^*)}) & \text{if} \quad N(L) \supset N_1(L) \cong N(L^*) \end{cases}$$

is a bounded right pseudoinverse of a normally solvable operator L.

Proof. Let us verify properties (11). For definiteness, let

$$N(L) \supset N_1(L) \cong N(L^*).$$

It follows from Lemma 1 that there exists a right inverse operator $\overline{L}_{r_0}^{-1}$.

First, we verify the third condition in (11). We have

$$LL_r^+ = L\overline{L}_{r_0}^{-1}(I_{\mathbf{H}_2} - P_{N(L^*)}) = (I_{\mathbf{H}_2} - \mathcal{P}_{N(L^*)})(I_{\mathbf{H}_2} - P_{N(L^*)})$$

$$= I_{\mathbf{H}_2} - \mathcal{P}_{N(L^*)} - P_{N(L^*)} + \mathcal{P}_{N(L^*)}P_{N(L^*)} = I_{\mathbf{H}_2} - P_{N(L^*)}$$

$$= (I_{\mathbf{H}_2} - P_{N(L^*)})^* = (LL_r^+)^*$$

because $\mathcal{P}_{N(L^*)}P_{N(L^*)} = \mathcal{P}_{N(L^*)}$ by virtue of the fourth relation in (9).

Further, we verify the first and the second condition in (11). We have

$$LL_r^+L = (I_{\mathbf{H}_2} - P_{N(L^*)})L = L - P_{N(L^*)}L = L$$

because $P_{N(L^*)}L = 0$, and

$$L_r^+LL_r^+ = L_r^+(I_{\mathbf{H}_2} - P_{N(L^*)}) = L_r^+ - \overline{L}_{r_0}^{-1}(I_{\mathbf{H}_2} - P_{N(L^*)})P_{N(L^*)}$$

$$= L_r^+ - \overline{L}_{r_0}^{-1}(P_{N(L^*)} - P_{N(L^*)}) = L_r^+$$

because $P_{N(L^*)}^2 = P_{N(L^*)}$.

It is easy to verify that the fourth condition in (2) is not satisfied. Indeed,

$$L_r^+L = \overline{L}_{r_0}^{-1}(I_{\mathbf{H}_2} - P_{N(L^*)})L = \overline{L}_{r_0}^{-1}L = I_{\mathbf{H}_1} - \mathcal{P}_{N(L)}$$

because, by virtue of Lemma 5, we have $\overline{L}_{r_0}^{-1}L = I_{\mathbf{H}_1} - \mathcal{P}_{N(L)}$, where $\mathcal{P}_{N(L)}\colon \mathbf{H}_1 \to N(L)$ is a projector.

The boundedness of the right pseudoinverse operator L_r^+ follows from the boundedness of the operators $\overline{L}_{l_0,r_0}^{-1}$ and $I_{\mathbf{H}_2} - P_{N(L^*)}$.

Theorem 2. *The operator*

$$L_l^+ = (I_{\mathbf{H}_1} - P_{N(L)})\overline{L}_{l_0,r_0}^{-1} = \begin{cases} (I_{\mathbf{H}_1} - P_{N(L)})\overline{L}_{l_0}^{-1} & \textit{if} \quad N(L) \cong N_1(L^*) \subset N(L^*), \\ (I_{\mathbf{H}_1} - P_{N(L)})\overline{L}_{r_0}^{-1} & \textit{if} \quad N(L) \supset N_1(L) \cong N(L^*) \end{cases}$$

is a left pseudoinverse of a normally solvable operator L.

Proof. The proof of Theorem 2 is analogous to the proof of Theorem 1.

Pseudoinverse Operator for a Linear, Bounded, Normally Solvable Operator

Using Theorems 1 and 2, we can obtain a formula for a pseudoinverse of a normally solvable operator in a Hilbert space.

Theorem 3. *The operator*

$$L^+ = L_l^+(I_{\mathbf{H}_2} - P_{N(L^*)}) = (I_{\mathbf{H}_1} - P_{N(L)})L_r^+ \tag{13}$$

is the unique bounded operator pseudoinverse to a normally solvable operator L.

Proof. Let us verify properties (2), which define a pseudoinverse operator. For definiteness, let $N(L) \cong N_1(L^*) \subset N(L^*)$. By virtue of Lemma 1, this implies that there exists a left inverse operator $\overline{L}_{l_0}^{-1}$.

Since

$$L(I_{\mathbf{H}_1} - P_{N(L)}) = L, \quad (I_{\mathbf{H}_2} - P_{N(L^*)})L = L, \quad \mathcal{P}_{N(L^*)}L = 0,$$

we obtain

$$LL^+L = L(I_{\mathbf{H}_1} - P_{N(L)})\overline{L}_{l_0}^{-1}(I_{\mathbf{H}_2} - P_{N(L^*)})L = L\overline{L}_{l_0}^{-1}L = (I_{\mathbf{H}_2} - \mathcal{P}_{N(L^*)})L = L,$$

which proves the first property.

Taking into account that $L\overline{L}_{l_0}^{-1} = I_{\mathbf{H}_2} - \mathcal{P}_{N(L^*)}$ by virtue of Lemma 5, and $\mathcal{P}_{N(L^*)}P_{N(L^*)} = \mathcal{P}_{N(L^*)}$ by virtue of Lemma 4, we get

$$L^+LL^+ = L^+L(I_{\mathbf{H}_1} - P_{N(L)})\overline{L}_{l_0}^{-1}(I_{\mathbf{H}_2} - P_{N(L^*)}) = L^+L\overline{L}_{l_0}^{-1}(I_{\mathbf{H}_2} - P_{N(L^*)})$$

$$= L^+(I_{\mathbf{H}_2} - \mathcal{P}_{N(L^*)})(I_{\mathbf{H}_2} - P_{N(L^*)}) = L^+(I_{\mathbf{H}_2} - \mathcal{P}_{N(L^*)} - P_{N(L^*)} + \mathcal{P}_{N(L^*)}) = L^+,$$

which proves the second property.

Let us verify the third and the fourth property. We have

$$LL^+ = L(I_{\mathbf{H}_1} - P_{N(L)})L_{l_0}^{-1}(I_{\mathbf{H}_2} - P_{N(L^*)}) = LL_{l_0}^{-1}(I_{\mathbf{H}_2} - P_{N(L^*)})$$

$$= (I_{\mathbf{H}_2} - \mathcal{P}_{N(L^*)})(I_{\mathbf{H}_2} - P_{N(L^*)}) = I_{\mathbf{H}_2} - P_{N(L^*)} = (LL^+)^*,$$

$$L^+L = L_l^+(I_{\mathbf{H}_1} - P_{N(L^*)})L = L_l^+L = I_{\mathbf{H}_1} - P_{N(L)} = (L^+L)^*$$

because L_l^+ is a left pseudoinverse operator.

The boundedness of the pseudoinverse operator L^+ follows from the boundedness of the one-sided pseudoinverse operators L_r^+ and L_l^+ and orthoprojectors $I_{\mathbf{H}_1} - P_{N(L)}$ and $I_{\mathbf{H}_2} - P_{N(L^*)}$.

The theorem is proved.

Remark 4. If $L^-: \mathbf{H}_2 \to \mathbf{H}_1$ is a generalized inverse operator that possesses properties (1), then, using a formula analogous to (13), we can prove that the operator

$$L^+ = (I_{\mathbf{H}_1} - P_{N(L)})L^-(I_{\mathbf{H}_2} - P_{N(L^*)})$$

is the unique pseudoinverse of a normally solvable operator L.

Using the formula for the pseudoinverse operator L^+ proposed in Theorem 3, we can find an explicit formula for a general solution of the linear operator equation (4) with a linear, bounded, normally solvable operator L.

Theorem 4. *Let $L: \mathbf{H}_1 \to \mathbf{H}_2$ be a normally solvable operator. Equation (4) is solvable for those and only those $y \in \mathbf{H}_2$ for which*

$$P_{N(L^*)}y = 0. \tag{14}$$

In this case, Eq. (4) has a family of solutions representable in the form of the direct orthogonal sum

$$x = \tilde{x} + \bar{x} = P_{N(L)}\hat{x} + L^+y, \tag{15}$$

where $\tilde{x}$ is a general solution of the corresponding homogeneous equation $Lx = 0$, $\bar{x}$ is the unique particular solution of the inhomogeneous operator equation (4), and $\hat{x}$ is an arbitrary element of the space $\mathbf{H}_1$.

Proof. It follows from (3) that a general solution of Eq. (4) is the direct orthogonal sum of a general solution $\tilde{x}$ of the homogeneous equation $Lx = 0$ corresponding to Eq. (4) and the unique particular solution $\bar{x} = L^+y$ of the inhomogeneous equation (4). It follows from the definition of an orthoprojector to the null space $N(L)$ of an operator L that a general solution of the homogeneous equation $Lx = 0$ can be represented in the form

$$\tilde{x} = P_{N(L)}\hat{x}.$$

Since the linear operator equation (4) is normally solvable, it is necessary and sufficient for its solvability [8] that y be orthogonal to any vector from the null space $N(L^*)$ of the adjoint operator L^*. Since $R(L) = N(\mathcal{P}_{N(L^*)})$ and $R(L)$ and $N(L^*)$ are mutually orthogonal and complement one another in the space $\mathbf{H}_2$, this condition is equivalent to condition (14), which guarantees that the element y belongs to the image $R(L)$ of the operator L.

Substituting solution (15) into the original equation (4) and taking into account the third relation in (2) and condition (14), we obtain

$$Lx = LP_{N(L)}\hat{x} + LL^+y = LL^+y = (I_{\mathbf{H}_2} - P_{N(L^*)})\,y = I_{\mathbf{H}_2}y - P_{N(L^*)}y = I_{\mathbf{H}_2}y = y$$

because $LP_{N(L)} = 0$.

The theorem is proved.

If condition (14) is not satisfied, i.e., y does not belong to the image $R(L)$ of the operator L, then the operator equation (4) does not have a solution. In this case, problem (4) is ill posed and has a so-called pseudosolution [12], which minimizes the norm of the residual $\|Lx - y\|_{\mathbf{H}_2}$.

Example 1. Let us find solvability conditions and the general form of a solution for the operator equation

$$Qx = y, \tag{16}$$

where the linear matrix operator

$$Q = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & -1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & -1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

acts from the real Hilbert space $\mathbf{l}^2$ of number sequences $x = \operatorname{col}(\xi^{(1)}, \xi^{(2)}, \xi^{(3)}, \dots, \xi^{(i)}, \dots)$ for which

$$\sum_{i=1}^{\infty} (\xi^{(i)})^2 < \infty$$

into the real Hilbert space $\mathbf{l}^2$ of number sequences $y = \operatorname{col}(\eta^{(1)}, \eta^{(2)}, \eta^{(3)}, \dots, \eta^{(j)}, \dots)$ for which

$$\sum_{j=1}^{\infty} (\eta^{(j)})^2 < \infty.$$

Let us verify the boundedness of the operator Q in this space. We have

$$\|Q\|_{\mathbf{l}^2} = \sup_{x \in \mathbf{l}^2, x \neq 0} \frac{\|Qx\|_{\mathbf{l}^2}}{\|x\|_{\mathbf{l}^2}} = \sup_{x \in \mathbf{l}^2, x \neq 0} \frac{\sup\limits_{j \in N} |\eta^{(j)}|}{\sup\limits_{i \in N} |\xi^{(i)}|}$$

$$= \sup_{x \in \mathbf{l}^2, x \neq 0} \frac{\sup\limits_{j \in N} (|\xi^{(1)} - \xi^{(2)}|, 0, |\xi^{(3)} - \xi^{(4)}|, 0, \dots)}{\sup\limits_{i \in N} |\xi^{(i)}|}$$

$$\leq \sup_{x \in \mathbf{l}^2, x \neq 0} \frac{\sup\limits_{j \in N} (|\xi^{(1)}| + |\xi^{(2)}|, 0, |\xi^{(3)}| + |\xi^{(4)}|, 0, \dots)}{\sup\limits_{i \in N} |\xi^{(i)}|} \leq 2 \frac{\sup\limits_{j \in N} |\xi^{(j)}|}{\sup\limits_{i \in N} |\xi^{(i)}|} = 2$$

because

$$\sup_{i\in N}(|\xi^{(i)}|+|\xi^{(i+1)}|)\le 2\sup_{i\in N}(|\xi^{(i)}|,|\xi^{(i+1)}|).$$

Thus, the operator $Q:\mathbf{l}^2\to\mathbf{l}^2$ is bounded.

The orthoprojectors $P_{N(Q)}$ and $P_{N(Q^*)}$ have the form

$$P_{N(Q)}=\operatorname{diag}\left\{\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix},\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix},\ldots\right\},$$

$$P_{N(Q^*)}=\operatorname{diag}\left\{\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},\ldots\right\}.$$

The pseudoinverse operator Q^+ has the form

$$Q^+=\operatorname{diag}\left\{\begin{pmatrix}\frac{1}{2} & 0\\ -\frac{1}{2} & 0\end{pmatrix},\begin{pmatrix}\frac{1}{2} & 0\\ -\frac{1}{2} & 0\end{pmatrix},\ldots\right\}.$$

By virtue of Theorem 4, the operator equation (16) has a bounded solution for those and only those

$$y\in\mathbf{l}^2,\quad y=\operatorname{col}(\eta^{(1)},\eta^{(2)},\eta^{(3)},\ldots),$$

for which

$$\mathcal{P}_{N(Q^*)}y=\begin{pmatrix}0 & 0 & 0 & 0 & \ldots\\ 0 & 1 & 0 & 0 & \ldots\\ 0 & 0 & 0 & 0 & \ldots\\ 0 & 0 & 0 & 1 & \ldots\\ \ldots & \ldots & \ldots & \ldots & \ldots\end{pmatrix}\begin{pmatrix}\eta^{(1)}\\ \eta^{(2)}\\ \eta^{(3)}\\ \eta^{(4)}\\ \ldots\\ \eta^{(i)}\\ \ldots\end{pmatrix}=\begin{pmatrix}0\\ \eta^{(2)}\\ 0\\ \eta^{(4)}\\ \ldots\\ \eta^{(2k)}\\ \ldots\end{pmatrix}=0. \tag{17}$$

The solvability condition (17) is satisfied, e.g., for

$$y\in\mathbf{l}^2,\quad y=\operatorname{col}(\eta^{(1)},0,\eta^{(3)},0,\eta^{(5)},0,\ldots).$$

For these vectors, the operator equation (16) has a solution $x\in\mathbf{l}^2$ bounded on $R(Q)$ of the following form:

$$x = P_{N(Q)}\hat{x} + Q^{+}y$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \dots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} \hat{\xi}^{(1)} \\ \hat{\xi}^{(2)} \\ \hat{\xi}^{(3)} \\ \hat{\xi}^{(4)} \\ \dots \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \dots \\ -\frac{1}{2} & 0 & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{2} & 0 & \dots \\ 0 & 0 & -\frac{1}{2} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} \eta^{(1)} \\ 0 \\ \eta^{(3)} \\ 0 \\ \dots \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}\,(\hat{\xi}^{(1)} + \hat{\xi}^{(2)} + \eta^{(1)}) \\ \frac{1}{2}\,(\hat{\xi}^{(1)} + \hat{\xi}^{(2)} - \eta^{(1)}) \\ \frac{1}{2}\,(\hat{\xi}^{(3)} + \hat{\xi}^{(4)} + \eta^{(3)}) \\ \frac{1}{2}\,(\hat{\xi}^{(3)} + \hat{\xi}^{(4)} - \eta^{(3)}) \\ \dots \end{pmatrix},$$

where $\hat{x} = \operatorname{col}(\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3, \dots, \hat{\xi}_i, \dots)$ is an arbitrary element of the Hilbert space $\mathbf{l}^2$, the vector

$$\operatorname{col}\left(\frac{1}{2}(\hat{\xi}^{(1)} + \hat{\xi}^{(2)}), \frac{1}{2}\,(\hat{\xi}^{(1)} + \hat{\xi}^{(2)}), \frac{1}{2}\,(\hat{\xi}^{(3)} + \hat{\xi}^{(4)}), \frac{1}{2}\,(\hat{\xi}^{(3)} + \hat{\xi}^{(4)}), \dots\right)$$

is a general solution of the homogeneous equation $Qx = 0$, and the vector

$$\operatorname{col}\left(\frac{\eta^{(1)}}{2}, -\frac{\eta^{(1)}}{2}, \frac{\eta^{(3)}}{2}, -\frac{\eta^{(3)}}{2}, \dots\right)$$

is the unique particular solution of the operator equation (16).

REFERENCES

1. I. Ts. Gokhberg and N. Ya. Krupnik, *Introduction to the Theory of One-Dimensional Singular Integral Operators* [in Russian], Shtiintsa, Kishinev (1973).
2. A. A. Boichuk and A. M. Samoilenko, *Generalized Inverse Operators and Fredholm Boundary-Value Problems,* VSP, Utrecht (2004).
3. E. H. Moore, "On the reciprocal of the general algebraic matrix (abstract)," *Bull. Amer. Math. Soc.,* No. 26, 394–395 (1920).
4. R. Penrose, "A generalized inverse for matrices," *Proc. Cambridge Phil. Soc.,* **51**, No. 3, 406–413 (1955).
5. Yu. P. Pyt'ev, "Pseudoinverse operator. Properties and applications," *Mat. Sb., Nov. Ser.,* **118**, No. 1, 19–49 (1982).
6. F. Riesz and B. Sz.-Nagy, *Leçons D'Analyse Fonctionnelle,* Akadémiai Kiadó, Budapest (1972).
7. V. A. Sadovnichii, *Theory of Operators* [in Russian], Moscow University, Moscow (1986).

8. V. A. Trenogin, *Functional Analysis* [in Russian], Nauka, Moscow (1980).
9. A. Sobczyk, "Projections in Minkowski and Banach spaces," *Duke Math. J.,* No. 8, 78–106 (1941).
10. A. A. Boichuk, V. F. Zhuravlev, and A. M. Samoilenko, *Generalized Inverse Operators and Noetherian Boundary-Value Problems* [in Russian], Institute of Mathematics, Ukrainian National Academy of Sciences, Kiev (1995).
11. V. S. Korolyuk and A. F. Turbin, *Mathematical Foundations of Phase Lumping of Compound Systems* [in Russian], Naukova Dumka, Kiev (1978).
12. F. R. Gantmakher, *Theory of Matrices* [in Russian], Nauka, Moscow (1988).