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GENERALIZATION OF THE SCHMIDT LEMMA TO THE CASE OF *n*-NORMAL AND *d*-NORMAL OPERATORS IN A BANACH SPACE

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We generalize the known Schmidt lemma to the case of linear, bounded, normally solvable operators that are n-normal or d-normal in infinite-dimensional Banach spaces. It is assumed that the kernels and images of these operators have complements in these spaces.

The Schmidt lemma [1] is most completely studied and widely used for the generalized inversion of linear, bounded, normally solvable Fredholm operators (with nonzero kernels) in the form of the so-called Schmidt construction [2]. Its analog for Noetherian operators in finite-dimensional Banach and Hilbert spaces was considered in [3].

The aim of the present paper is to prove statements that generalize the Schmidt lemma to the case of bounded normally extendable operators that are n-normal or d-normal and act in infinite-dimensional Banach spaces.

Statement of the Problem

Let L be a linear, bounded, normally solvable operator that acts from a Banach space $\mathbf{B}_1$ into a Banach space $\mathbf{B}_2$. Denote the dimensions of the null spaces of the operator L and its adjoint L^* by $\dim N(L) = \mu$ and $\dim N(L^*) = \nu$, respectively. According to S. Krein's classification [4], a normally solvable operator L is n-normal if μ is finite and ν is infinite, and it is d-normal if μ is infinite and ν is finite.

If $L\colon \mathbf{B}_1 \to \mathbf{B}_2$ is a linear bounded n-normal operator, then we assume that its image $R(L)$ has a complement in the space $\mathbf{B}_2$ [5], i.e.,

$$B_2 = Y \oplus R(L), \tag{1}$$

and if $L\colon \mathbf{B}_1 \to \mathbf{B}_2$ is a linear bounded d-normal operator, then its kernel $N(L)$ has a complement in the space $\mathbf{B}_1$, i.e.,

$$B_1 = N(L) \oplus X. \tag{2}$$

Main Result

First, we consider n-normal operators. By virtue of its finite dimensionality $(\mu < \infty)$, the subspace $N(L)$ has a complete system of basis elements $\{f_i\}_{i=1}^{\mu} \subset N(L)$, $f_i = \operatorname{col}(f_i^{(1)}, f_i^{(2)}, f_i^{(3)}, \ldots)$. Assume that the space $\mathbf{B}_2$ has a basis. It is known [6, p. 131] that $\mathbf{B}_2^*$ also has a basis. Therefore, the subspace $N^*(L) \subset \mathbf{B}_2^*$ has a complete system of basis elements (functionals) $\{\varphi_s(\cdot)\}_{s=1}^{\infty} \subset N(L^*)$, $\varphi_s(\cdot) = \operatorname{col}(\varphi_s^{(1)}(\cdot), \varphi_s^{(2)}(\cdot), \varphi_s^{(3)}(\cdot), \ldots)$. For the elements $\{f_i\}_{i=1}^{\mu}$ and functionals $\{\varphi_s(\cdot)\}_{s=1}^{\infty}$, there exist an adjoint biorthogonal [7] system of functionals $\{\gamma_j(\cdot)\}_{j=1}^{\mu} \subset \mathbf{B}_1^*$, $\gamma_j(\cdot) = \operatorname{col}(\gamma_j^{(1)}(\cdot), \gamma_j^{(2)}(\cdot), \gamma_j^{(3)}(\cdot), \ldots)$, and an adjoint biorthogonal complete system of

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elements $\{\psi_k\}_{k=1}^{\infty} \subset \mathbf{B}_2$, $\psi_k = \text{col}(\psi_k^{(1)}, \psi_k^{(2)}, \psi_k^{(3)}, \ldots)$. Note that, according to the Hahn–Banach theorem, each functional $\{\gamma_j(\cdot)\}_{j=1}^{\mu}$ defined on the subspace $N(L) \subset \mathbf{B}_1$ can be extended, with preservation of norm, to the entire space $\mathbf{B}_1$.

Let

$$X = (f_1, f_2, \ldots, f_\mu), \quad \Gamma(\cdot) = (\gamma_1(\cdot), \gamma_2(\cdot), \ldots, \gamma_\mu(\cdot))^T$$
$$\Phi(\cdot) = (\varphi_1(\cdot), \varphi_2(\cdot), \ldots, \varphi_k(\cdot), \ldots)^T, \quad \Psi = (\psi_1, \psi_2, \ldots, \psi_k, \ldots) \tag{3}$$

denote, respectively, $\infty\times\mu$, $\mu\times\infty$, $\infty\times\infty$, and $\infty\times\infty$ matrices; furthermore, $\Gamma(X) = E_\mu$ and $\Phi(\Psi) = E_\infty$, where E_μ and E_∞ are the identity matrices.

We construct a projection operator $\mathcal{P}_{N(L)}\colon \mathbf{B}_1 \to N(L)$ according to the formula

$$\mathcal{P}_{N(L)}(\cdot) = X\Gamma(\cdot), \quad \mathcal{P}_{N(L)}\colon B_1 \to B_1.$$

To construct a projection operator $\mathcal{P}_Y\colon B_2 \to B_2$, we define the sequence of projectors

$$\mathcal{P}_{Y(j)}(\cdot) = \Psi_j \Phi_j(\cdot) \tag{4}$$

of the space $\mathbf{B}_2$ to the subspaces $Y_j \subset Y$ spanned by the elements $\{\psi_k\}_{k=1}^{j}$.

Lemma 1. *The sequence (4) of projectors $\mathcal{P}_{Y(j)}$ converges strongly (pointwise) to the projector*

$$\mathcal{P}_Y(\cdot) = \Psi\Phi(\cdot) = \lim_{j\to\infty} \Psi_j \Phi_j(\cdot), \quad \mathcal{P}_Y\colon B_2 \to Y,$$

where $Y \subset \mathbf{B}_2$ is an infinite-dimensional space spanned by the complete system of elements $\{\psi_s\}_{s=1}^{\infty}$.

Proof. According to the definition of strong convergence in the norm of the space $\mathbf{B}_2$, with regard for the definition of the matrices Φ and Ψ we get

$$\|\mathcal{P}_Y y - \mathcal{P}_{Y_j} y\| = \left\| \sum_{\xi=1}^{\infty} \varphi_\xi(y)\psi_\xi - \sum_{\xi=1}^{j} \varphi_\xi(y)\psi_\xi \right\|$$

$$= \left\| \sum_{\xi=j+1}^{\infty} \varphi_\xi(y)\psi_\xi \right\| \le \sum_{\xi=j+1}^{\infty} \|\varphi_\xi(y)\psi_\xi\| \quad \forall y \in Y \subset \mathbf{B}_2.$$

The quantity

$$\sum_{\xi=j+1}^{\infty} \|\varphi_\xi(y)\psi_\xi\|$$

tends to zero as $j \to \infty$ as a remainder of the expansion

$$\sum_{\xi=1}^{\infty} \varphi_\xi(y)\psi_\xi$$

of an element $y \in Y$ in the system of elements $\{\psi_\xi\}_{\xi=1}^{\infty}$. Since the functionals $\{\varphi_j(\cdot)\}_{j=1}^{\infty}$ can be extended to the entire space $\mathbf{B}_2$ with preservation of norm, we can conclude that

$$\sum_{\xi=j+1}^{\infty} \|\varphi_\xi(y)\psi_\xi\| \to 0 \quad \text{as} \quad j \to \infty$$

for any $y \in \mathbf{B}_2$.

The lemma is proved.

Let us show that the constructed projectors divide the spaces $\mathbf{B}_1$ and $\mathbf{B}_2$ into mutually complementary subspaces according to relations (1) and (2).

Lemma 2. *The operators $\mathcal{P}_{N(L)}$ and $\mathcal{P}_Y$ are bounded projectors in the Banach spaces $\mathbf{B}_1$ and $\mathbf{B}_2$ and divide these spaces into direct sums of closed subspaces according to relations (1) and (2).*

Proof. First, we prove that the operators $\mathcal{P}_{N(L)}$ and $\mathcal{P}_Y$ are projectors, i.e., that they satisfy the conditions $\mathcal{P}^2_{N(L)} = \mathcal{P}_{N(L)}$ and $\mathcal{P}^2_Y = \mathcal{P}_Y$. Indeed, we have

$$\mathcal{P}^2_{N(L)}(\cdot) = \mathcal{P}_{N(L)}(\mathcal{P}_{N(L)}(\cdot)) = X\Gamma(X\Gamma(\cdot)) = X\Gamma(X)\Gamma(\cdot) = X\Gamma(\cdot) = \mathcal{P}_{N(L)}(\cdot)$$

because $\Gamma(X) = E_\mu$, and

$$\mathcal{P}^2_Y(\cdot) = \mathcal{P}_Y(\mathcal{P}_Y(\cdot)) = \Psi\Phi(\Psi\Phi(\cdot)) = \Psi\Phi(\Psi)\Phi(\cdot) = \Psi\Phi(y\cdot) = \mathcal{P}_Y(\cdot)$$

because $\Phi(\Psi) = E_\nu$.

Thus, the projectors $\mathcal{P}_{N(L)}$ and $\mathcal{P}_Y$ divide the spaces $\mathbf{B}_1$ and $\mathbf{B}_2$ into direct topological sums of closed subspaces:

$$\mathbf{B}_1 = N(\mathcal{P}_{N(L)}) \oplus R(\mathcal{P}_{N(L)}), \quad \mathbf{B}_2 = N(\mathcal{P}_Y) \oplus R(\mathcal{P}_Y).$$

Further, we show that

$$N(L) = R(\mathcal{P}_{N(L)}), \quad R(L) = N(\mathcal{P}_Y),$$
$$Y = R(\mathcal{P}_Y), \quad X = N(\mathcal{P}_{N(L)}). \tag{5}$$

Since $L\mathcal{P}_{N(L)}x = LX\Gamma(x) = 0$, $x \in \mathbf{B}_1$, we have $R(\mathcal{P}_{N(L)}) \subset N(L)$. Let $x \in N(L)$. Then $x = Xc$. Applying the matrix of functionals Γ to the last equality, we get $c = \Gamma(x)$, i.e., $x = X\Gamma(x)$. Therefore, $x = \mathcal{P}_{N(L)}x$ and $x \in R(\mathcal{P}_{N(L)})$. Thus, $N(L) \subset R(\mathcal{P}_{N(L)})$, and the first equality in (5) is proved.

Since $\mathcal{P}_Y Lx = \Psi\Phi(Lz) = \Psi(L^*\Phi)(z) = 0$ (φ_s are basis vectors of the null space of the operator L^*), we have $R(L) \subset N(\mathcal{P}_Y)$. On the other hand, if $y \in N(\mathcal{P}_Y)$, then

$$\mathcal{P}_Y y = \Psi\Phi(y) = 0,$$

i.e., $\varphi_s(y) = 0$, $s = 1, 2, \ldots, \infty$. By virtue of the normal solvability of the operator L, this means that $y \in R(L)$. Therefore, $N(\mathcal{P}_Y) \subset R(L)$, and the proof of the second equality in (5) is completed.

The third and the fourth equality in (5) are proved by analogy.

Thus, the projectors $\mathcal{P}_{N(L)}$ and $\mathcal{P}_Y$ divide the Banach spaces B_1 and B_2 into direct sums of closed subspaces according to relations (1) and (2).

The boundedness of the projector $\mathcal{P}_{N(L)}$ follows from its finite dimensionality, and the boundedness of the projector $\mathcal{P}_Y$ follows from the complementability of the image $R(L)$ of the operator L [8].

The lemma is proved.

Since the system of basis elements $\{\varphi(\cdot)_s\}_{s=1}^{\nu} \subset B_2^*$ of the null space $N(L^*)$ and the system of elements $\{\psi_s\}_{s=1}^{\nu} \subset Y \subset B_2$ are adjoint biorthogonal, $\varphi_s(\psi_k) = \delta_{sk}$, there exists a one-to-one correspondence between them. Therefore, the subspaces $N(L^*)$ and Y are isomorphic and have the same dimension: $\dim N(L^*) = \dim Y$. Since μ is finite and ν is infinite, we can establish an isomorphism between $N(L)$ and a certain subspace $Y_1 \subset Y$.

We now construct this isomorphism.

Let

$$\overline{\Phi}(\cdot) = (\overline{\varphi}_1(\cdot), \overline{\varphi}_2(\cdot), \ldots, \overline{\varphi}_\mu(\cdot))^T \quad \text{and} \quad \overline{\Psi} = (\overline{\psi}_1, \overline{\psi}_2, \ldots, \overline{\psi}_\mu) \tag{6}$$

denote, respectively, $\mu \times \infty$ and $\infty \times \mu$ matrices composed of μ rows and columns of the matrices Φ and Ψ, respectively. The matrix $\overline{\Psi}$ is composed of the system of elements $\{\overline{\psi}_k\}_{k=1}^{\nu} \subset \{\psi_k\}_{k=1}^{\infty}$ spanning the subspace Y_1. The matrix $\overline{\Phi}$ is composed of functionals $\{\overline{\varphi}_s\}_{s=1}^{\nu} \subset \{\varphi_s\}_{s=1}^{\infty}$ that satisfy the relation $\overline{\Phi}(\overline{\Psi}) = E_\mu$. We construct a linear, bounded, invertible operator $J: N(L) \to Y_1 \subseteq Y$ that performs an isomorphism of $N(L)$ onto Y_1 and its inverse $J^{-1}: Y_1 \to N(L)$ according to the relations

$$J(\cdot) = \overline{\Psi}\,\Gamma(\cdot), \quad (\cdot) \in N(L),$$

$$J^{-1}(\cdot) = X\,\overline{\Phi}(\cdot), \quad (\cdot) \in Y_1.$$

By virtue of the Hahn–Banach theorem, each linear functional γ_i can be extended to the entire space $\mathbf{B}_1$ with preservation of norm, and each linear functional $\overline{\varphi}_s$ can be extended to the entire space $\mathbf{B}_2$. In this connection, we denote the extension of the operator $J: N(L) \to Y$ to the entire space $\mathbf{B}_1$ by $\overline{\mathcal{P}}_{Y_1}$ and the extension of its inverse J^{-1} to the space $\mathbf{B}_2$ by $\overline{\mathcal{P}}_{N(L)}$, i.e.,

$$\overline{\mathcal{P}}_{Y_1}(\cdot) = \overline{\Psi}\Gamma(\cdot), \quad (\cdot) \in B_1,$$

$$\overline{\mathcal{P}}_{N(L)}(\cdot) = X\overline{\Phi}(\cdot), \quad (\cdot) \in B_2.$$

Using (6), we define the projector $\mathcal{P}_{Y_1}: \mathbf{B}_2 \to Y_1 \subset Y$ as follows:

$$\mathcal{P}_{Y_1}(\cdot) = \overline{\Psi}\,\overline{\Phi}(\cdot).$$

This operator divides the subspace Y into a direct topological sum of subspaces, namely,

$$Y = Y_1 \oplus Y_2, \tag{7}$$

where $Y_2 = \mathcal{P}_{Y_2}\mathbf{B}_2 = (\mathcal{P}_Y - \mathcal{P}_{Y_1})\mathbf{B}_2$, and is bounded.

For the class of normally solvable n-normal operators, we prove the following statement, which is an analog of the Schmidt lemma:

Lemma 3. *Let* $L\colon \mathbf{B_1} \to \mathbf{B_2}$ *be a linear bounded* n*-normal operator and let the image* $R(L)$ *have a complement in the space* $\mathbf{B_2}$*. Then the operator* $\overline{L} = L + \overline{\mathcal{P}}_{Y_1}$ *has a bounded left-inverse operator:*

$$\overline{L}_{l_0}^{-1} = (L + \overline{\mathcal{P}}_{Y_1})_l^{-1}.$$

The general form of the left-inverse operators $\overline{L}_{l_0}^{-1}$ *is given by the relation*

$$\overline{L}_{l_0}^{-1} = \overline{L}_{l_0}^{-1}(I_{\mathbf{B}_2} - \mathcal{P}_{Y_2}).$$

Proof. Let L be an n-normal operator. For the operator $\overline{L}$ to be left invertible, it is necessary and sufficient that the following conditions be satisfied [9]:

(a) $\ker \overline{L} = \{0\}$;

(b) the linear manifold $R(\overline{L})$ is a subspace that has a direct complement in $\mathbf{B_2}$.

Let us show that $\ker \overline{L} = \{0\}$. Assume that there exists $x_0 \neq 0$, $x_0 \in \mathbf{B_1}$, such that

$$(L + \overline{\mathcal{P}}_{Y_1})x_0 = Lx_0 + \overline{\Psi}\,\Gamma(x_0) = 0.$$

It is obvious that $Lx_0 \in R(L)$. It follows from the definition of $\overline{\mathcal{P}}_{Y_1}$ that $\overline{\mathcal{P}}_{Y_1}x_0 \in Y_1 \subset Y$. Since the subspaces $R(L)$ and Y mutually complement one another to the entire space $\mathbf{B_2}$, we have $R(L)\bigcap Y = \{0\}$, i.e., they have only one common element, namely the zero element. Thus, $Lx_0 = 0$ and $\overline{\mathcal{P}}_{Y_1}x_0 = 0$. This implies that $x_0 \in N(L)$ and $x_0 \in N(\overline{\mathcal{P}}_{Y_1}) \subset X$. Since the subspaces $N(L)$ and X also mutually complement one another to the space $\mathbf{B_1}$, we have $N(L)\bigcap X = \{0\}$. This yields $x_0 = 0$.

The complementability of the image $R(\overline{L})$ in the space $\mathbf{B_2}$ follows from relation (7) and the complementability of the subspace $R(L)$:

$$\mathbf{B_2} = R(L) \oplus Y_1 \oplus Y_2 = R(\overline{L}) \oplus Y_2. \tag{8}$$

Therefore, the operator $\overline{L}$ has a left inverse. Since the operator $\overline{L}$ maps the Banach space $\mathbf{B_1}$ bijectively to the subspace $\mathbf{B_2} \ominus Y_2$, it follows from the Banach theorem [10] that the operator $\overline{L}_l^{-1}$ is bounded. It is known [9, p. 61] that if the projection operator $\mathcal{P}$ possesses the property $R(\mathcal{P}) = R(\overline{L})$, then the general form of left-inverse operators admits the representation $\overline{L}_{l_0}^{-1}\mathcal{P}$. It follows from (8) that the operator $I_{\mathbf{B}_2} - \mathcal{P}_{Y_2}$ possesses this property, i.e., $R(I_{\mathbf{B}_2} - \mathcal{P}_{Y_2}) = R(\overline{L})$. Therefore, the general representation of left-inverse operators can be rewritten as follows:

$$\overline{L}_{l_0}^{-1} = \overline{L}_{l_0}^{-1}(I_{\mathbf{B}_2} - \mathcal{P}_{Y_2}).$$

The lemma is proved.

Remark 1. If $\dim\ker L < \dim\ker L^* < \infty$, i.e., L is a Noetherian operator of negative index, then Lemma 3 reduces to Lemma 2.4 in [3, p. 47].

Remark 2. If $\dim\ker L = \dim\ker L^* = n < \infty$, i.e., L is a Fredholm operator of nonzero index, then Lemma 3 reduces to the Schmidt lemma [2, p. 340].

Now let $L\colon B_1 \to B_2$ be a linear bounded d-normal operator. In this case, the subspace $N(L)$ is infinite-dimensional ($\mu = \infty$) and the subspace $N(L^*)$ is finite-dimensional ($\nu < \infty$). Assume that the space $\mathbf{B}_1$ has a basis. Then $N(L)$ also has a basis. Let $\{f_i\}_{i=1}^{\infty} \subset N(L)$ be a complete system of basis elements. The subspace $N(L^*)$ has a finite-dimensional basis $\{\varphi_s\}_{s=1}^{\nu} \subset N(L^*)$. For the elements $\{f_i\}_{i=1}^{\infty}$ and functionals $\{\varphi_s\}_{s=1}^{\nu}$, there exist an adjoint biorthogonal system of functionals $\{\gamma_j\}_{j=1}^{\infty} \subset \mathbf{B}_1^*$ and an adjoint biorthogonal complete system of elements $\{\psi_k\}_{k=1}^{\nu} \subset \mathbf{B}_2$ [7]. Each of the functionals $\{\gamma_j\}_{j=1}^{\infty}$ and $\{\varphi_s\}_{s=1}^{\nu}$ defined on the subspaces $N(L) \subset \mathbf{B}_1$ and $Y \subset \mathbf{B}_2$, according to the Hahn–Banach theorem, can be extended to the spaces $\mathbf{B}_1$ and $\mathbf{B}_2$, respectively, with preservation of norm.

By analogy with (3), let

$$X = (f_1, f_2, \ldots, f_s, \ldots), \quad \Gamma(\cdot) = (\gamma_1(\cdot), \gamma_2(\cdot), \ldots, \gamma_s(\cdot), \ldots)^T,$$

$$\Phi(\cdot) = (\varphi_1(\cdot), \varphi_2(\cdot), \ldots, \varphi_\nu(\cdot))^T, \quad \Psi = (\psi_1, \psi_2, \ldots, \psi_\nu)$$

denote $\infty\times\infty$, $\infty\times\infty$, $\nu\times\infty$, and $\infty\times\nu$ matrices, respectively; furthermore, $\Gamma(X) = E_\infty$ and $\Phi(\Psi) = E_\nu$, where E_∞ and E_ν are the identity matrices.

To construct a projection operator $\mathcal{P}_{N(L)}\colon B_1 \to N(L)$, we define the sequence of projectors

$$\mathcal{P}_{N^{(i)}(L)}(\cdot) = X_i\Gamma_i(\cdot), \quad i = 1, 2, 3, \ldots, \tag{9}$$

of the space B_1 to the subspaces $N_i(L)$ of the null space $N(L)$.

Lemma 4. *The sequence (9) of projectors $\mathcal{P}_{N^{(i)}(L)}$ converges strongly (pointwise) to the projector*

$$\mathcal{P}_{N(L)}(\cdot) = X\Gamma(\cdot) = \lim_{i\to\infty} X_i\Gamma_i(\cdot), \quad \mathcal{P}_{N(L)}\colon B_1 \to N(L). \tag{10}$$

Proof. The proof is analogous to the proof of Lemma 1.

We define a projection operator $\mathcal{P}_Y\colon \mathbf{B}_2 \to Y$ of the space $\mathbf{B}_2$ to the subspace Y as follows:

$$\mathcal{P}_Y(\cdot) = \Psi\Phi(\cdot) \tag{11}$$

Note that, for the projection operators (10) and (11), Lemma 2 is true.

Since μ is infinite and ν is finite, we can establish an isomorphism between $N_1(L) \subset N(L)$ and Y.

We now construct this isomorphism. Let

$$\overline{X} = (\overline{f}_1, \overline{f}_2, \ldots, \overline{f}_\nu) \qquad \text{and} \qquad \overline{\Gamma}(\cdot) = (\overline{\gamma}_1(\cdot), \overline{\gamma}_2(\cdot), \ldots, \overline{\gamma}_\nu(\cdot))^T \tag{12}$$

denote, respectively, $\infty \times \nu$ and $\nu \times \infty$ matrices. Then we construct a linear, bounded, invertible operator $J\colon N_1(L) \to Y$ that realizes an isomorphism of $N_1(L)$ onto Y and its inverse $J^{-1}\colon Y \to N_1(L)$ as follows:

$$J(\cdot) = \Psi\,\overline{\Gamma}(\cdot), \quad (\cdot) \in N_1(L),$$

$$J^{-1}(\cdot) = \overline{X}\,\Phi(\cdot), \quad (\cdot) \in Y.$$

The matrix $\overline{X}$ is composed of ν columns of the matrix X, and the matrix $\overline{\Gamma}(\cdot)$ is composed of functionals of the matrix $\Gamma(\cdot)$ that satisfy the relation $\overline{\Gamma}(\overline{X}) = E_\nu$.

Let $\overline{\mathcal{P}}_Y$ denote an extension of the operator $J\colon N(L) \to Y$ to the entire space B_1 and let $\overline{\mathcal{P}}_{N_1(L)}$ denote an extension of its inverse J^{-1} to the space B_2, i.e.,

$$\overline{\mathcal{P}}_Y(\cdot) = \Psi\overline{\Gamma}(\cdot), \quad (\cdot) \in B_1,$$

$$\overline{\mathcal{P}}_{N_1(L)}(\cdot) = \overline{X}\Phi(\cdot), \quad (\cdot) \in B_2.$$

By analogy with (11), we define the projection operator $\mathcal{P}_{N_1(L)}\colon \mathbf{B}_1 \to N_1(L) \subset N(L)$ as follows:

$$\mathcal{P}_{N_1(L)}(\cdot) = \overline{X}\,\overline{\Gamma}(\cdot). \tag{13}$$

This operator is bounded and divides the subspace $N(L)$ into a direct topological sum of subspaces:

$$N(L) = N_1(L) \oplus N_2(L), \quad N_2(L) = \mathcal{P}_{N_2(L)}\mathbf{B}_1, \tag{14}$$

where $\mathcal{P}_{N_2(L)} = \mathcal{P}_{N(L)} - \mathcal{P}_{N_1(L)}$ is a bounded projector.

For the class of normally solvable d-normal operators, we prove a statement analogous to the Schmidt lemma.

Lemma 5. *Let $L\colon \mathbf{B_1} \to \mathbf{B_2}$ be a linear bounded d-normal operator and let the kernel $N(L)$ have a complement in the space $\mathbf{B_1}$. Then the operator $\overline{L} = L + \overline{\mathcal{P}}_Y$ has a bounded right-inverse operator:*

$$\overline{L}_{r_0}^{-1} = (L + \overline{\mathcal{P}}_Y)_r^{-1}.$$

The general form of the right-inverse operators $\overline{L}_{r_0}^{-1}$ is given by the relation

$$\overline{L}_{r_0}^{-1} = (I_{\mathbf{B}_1} - \mathcal{P}_{N_2(L)})\overline{L}_r^{-1}.$$

Proof. For the operator $\overline{L}$ to be right invertible, it is necessary and sufficient that the following conditions be satisfied [9]:

(a) $R(\overline{L}) = \mathbf{B}_2$;

(b) the subspace $N(\overline{L})$ has a direct complement in $\mathbf{B_1}$.

Using the second equality in (5), we get $R(L) = N(\mathcal{P}_Y)$, i.e., the condition $R(\overline{L}) = \mathbf{B_2}$ is equivalent to the condition

$$\mathcal{P}_Y(\cdot) = \Psi\Phi(\cdot) = 0.$$

Since the system of elements $\{\psi_s\}_{s=1}^{\nu}$ is linearly independent, the last relation holds if and only if all elements of $\{\varphi_s\}_{s=1}^{\nu}$ are equal to zero. This, in turn, means that the null space of the adjoint operator is trivial, i.e., $N(L^*) = \{0\}$.

Let us show that $N(\overline{L}^{\,*}) = \{0\}$. Assume that there exists a functional $\varphi_0, \ \varphi_0 \neq 0, \ \varphi \in \mathbf{B}_2^*$, such that $\overline{L}^{\,*}\varphi_0 = (L + \overline{\mathcal{P}}_Y)^*\varphi_0 = 0$. Taking into account the definition of the operator $\overline{\mathcal{P}}_Y$, we get

$$L^*\varphi_0 = -\overline{\mathcal{P}}_Y^{\,*}\varphi_0.$$

Applying the functionals $L^*\varphi_0 \in \mathbf{B}_1^*$ and $\overline{\mathcal{P}}_Y^{\,*}$ to the matrix X, we obtain, on the one hand,

$$(L^*\varphi_0)(X) = \varphi_0(LX) = 0$$

because $LX = 0$ and, on the other hand,

$$\overline{\mathcal{P}}_Y^{\,*}\varphi_0(X) = \varphi_0(\overline{\mathcal{P}}_Y X) = \varphi_0(\Psi)\overline{\Gamma}(X) = \varphi_0(\Psi)$$

because $\overline{\Gamma}(X) = \delta_{ij}$. Since the system of elements $\{\psi_i\}_{i=1}^{\nu}$ is linearly independent, the equality $\varphi_0(\Psi) = 0$ is possible only for $\varphi_0 = 0$. This contradiction proves that $N(\overline{L}^{\,*}) = \{0\}$, which, in turn, means that $R(\overline{L}) = \mathbf{B}_2$.

The complementability of the null space $N(\overline{L})$ follows from the definition of the projector $\mathcal{P}_{N_1(L)}$ (13) and the decomposition (14) of the null space $N(L)$ of the operator L.

It is known [9, p. 62] that if a projection operator $\mathcal{P}$ possesses the property $N(\mathcal{P}) = N(\overline{L})$, then the general form of right-inverse operators admits the representation $\mathcal{P}\,\overline{L}_{r_0}^{-1}$. It follows from (14) that the operator $I_{\mathbf{B}_1} - \mathcal{P}_{N_2(L)}$ possesses this property, i.e., $N(I_{\mathbf{B}_1} - \mathcal{P}_{N_2(L)}) = N(\overline{L})$. Therefore, the general representation of right-inverse operators can be rewritten as follows:

$$\overline{L}_{r_0}^{-1} = (I_{\mathbf{B}_1} - \mathcal{P}_{N_2(L)})\overline{L}_r^{-1}.$$

The lemma is proved.

Remark 3. If $\dim\ker L^* < \dim\ker L < \infty$, i.e., L is a Noetherian operator of positive index, then Lemma 5 reduces to Lemma 2.4 in [3, p. 47].

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