On coefficients of transitiveness of posets of special type

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For a quiver $Q = (Q_0, Q_1)$ with the set of vertices Q_0 and the set of arrows Q_1, P. Gabriel introduced the following quadratic form, called by him the quadratic Tits form of Q:

$$q_Q(z) = q_Q(z_1, \dots, z_n) := \sum_{i \in Q_0} z_i^2 - \sum_{i \to j} z_i z_j,$$

where $i \to j$ runs through the set Q_1. He proved that the quiver Q has finite representation type over a field k iff its Tits form is positive. This Gabriel's result laid the foundations of a new direction in the representation theory. This quadratic form is naturally generalized to a finite poset $S \not\ni 0$:

$$q_S(z) = z_0^2 + \sum_{i \in S} z_i^2 + \sum_{i<j, i,j \in S} z_i z_j - z_0 \sum_{i \in S} z_i.$$

Yu. A. Drozd showed that a poset S has finite representation type iff its Tits form is weakly positive (representations of posets were introduced by L. A. Nazarova and A. V. Roiter). For posets, in contrast to quivers, the sets of those with weakly positive and with positive Tits forms do not coincide. Therefore the investigations of posets with positive Tits form seems to be quite natural; they are analogs of the Dynkin diagrams. Posets of this type were classified in [1]. In this paper it is also introduced and classified the P-critical posets, which are analogs of the extended Dynkin diagrams. A poset S is called P-critical if its Tits quadratic form is not positive, but that of any proper subset of S is positive.

Let S be a finite poset and $S^2_< := \{(x, y) \mid x, y \in S, x < y\}$. If $(x, y) \in S^2_<$ and there is no z satisfying $x < z < y$, then we say that x and y are neighboring. We put $n_w = n_w(S) := |S^2_<|$ and denote by $n_e = n_e(S)$ the number of pairs of neighboring elements. On the language of the Hasse diagram $H(S)$, n_e is equal to the number of all its edges and n_w to the number of all its paths, up to parallelity, going bottom-up (two path is called parallel if they start and terminate at the same vertices). The ratio $k_t = k_t(S)$ of the numbers $n_w - n_e$ and n_w we call the coefficient of transitiveness of S. If $n_w = 0$ (then $n_e = 0$), we assume $k_t = 0$.

Recall that an element of a poset T is called nodal, if it is comparable with all elements of T. It follows from the results of [1] that any P-critical poset S is uniquely represented in the form $S = S_0^- 0 \cup S_1 \cup S_0^+$ where S_0^-, S_0^+ are chains (may be empty), S_1 does not contain nodal elements and $S_0^- < S_1 < S_0^+$ ($X < Y$ means that $x < y$ for any $x \in X$, $y \in Y$). Then $S_0 = S_0^- \cup S_0^+$ is the set of all nodal elements of S.

Theorem. *Let S be a P-critical poset. Then the following conditions are equivalent* :

a) $k_t(S) \geqslant k_t(T)$ *for any P-critical poset T;*

b) $|S_0| \geqslant |T_0|$ *for any P-critical poset T, and S_0^- or S_0^+ is empty.*

These studies were carried out together with Prof. V.M. Bondarenko.

1. Bondarenko V.M., Stepochkina M.V. (Min, max)-equivalence of partially ordered sets and the Tits quadratic form. Zb. Pr. Inst. Mat. NAN Ukr. / Problems of Analysis and Algebra /.—K.: Institute of Mathematics of NAN of Ukraine, 2005, vol. 2, no. 3, 18–58 (in Russian).