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# Coefficients of transitiveness of $P$-critical posets

Ми вводимо інваріант для будь-якої скінченної частково впорядкованої множини, який називаємо коефіцієнтом транзитивності, і обчислюємо його для всіх $P$-критичних частково впорядкованих множин, які є аналогом розширених діаграм Динкіна.

We introduce an invariant of a finite poset, called the coefficient of transitiveness, and calculate it for all $P$-critical posets, which are an analog of the extended Dynkin diagrams.

**1. Introduction.** In [1], for a finite quiver (directed graph) $Q$ with the set of vertices $Q_0$ and the set of arrows $Q_1$, P. Gabriel introduced a quadratic form $q_Q : \mathbb{Z}^n \to \mathbb{Z}$, $n = |Q_0|$, called by him the *quadratic Tits form of the quiver* $Q$:

$$q_Q(z) = q_Q(z_1, \ldots, z_n) := \sum_{i \in Q_0} z_i^2 - \sum_{i \to j} z_i z_j,$$

where $i \to j$ runs through the set $Q_1$. He proved that the quiver $Q$ has finite representation type over a field $k$ (i.e., finitely many indecomposable representations, up to isomorphism) if and only if its Tits form is positive. This Gabriel's work laid the foundations of a new direction in the theory of algebra dealing with the investigation of the relationships between the properties of representations of various objects and the properties of quadratic forms associated with these objects.

The above quadratic form is naturally generalized to a (finite) poset $S \not\ni 0$:

$$q_S(z) = z_0^2 + \sum_{i \in S} z_i^2 + \sum_{i<j, i,j \in S} z_i z_j - z_0 \sum_{i \in S} z_i.$$

In [2] Yu. A. Drozd showed that a poset $S$ has finite representation type if and only if its Tits form is weakly positive, i.e., takes positive value on any nonzero vector with nonnegative coordinates (representations of posets were introduced by L. A. Nazarova and A. V. Roiter in [3]).

For posets, in contrast to quivers, the sets of those with weakly positive and with positive Tits forms do not coincide. Therefore the investigations of posets with positive Tits form seems to be quite natural; notice that they are analogs of the Dynkin diagrams. Posets of this type were studied by the authors (from different points of view) in many papers (see e.g. [4] – [7]).

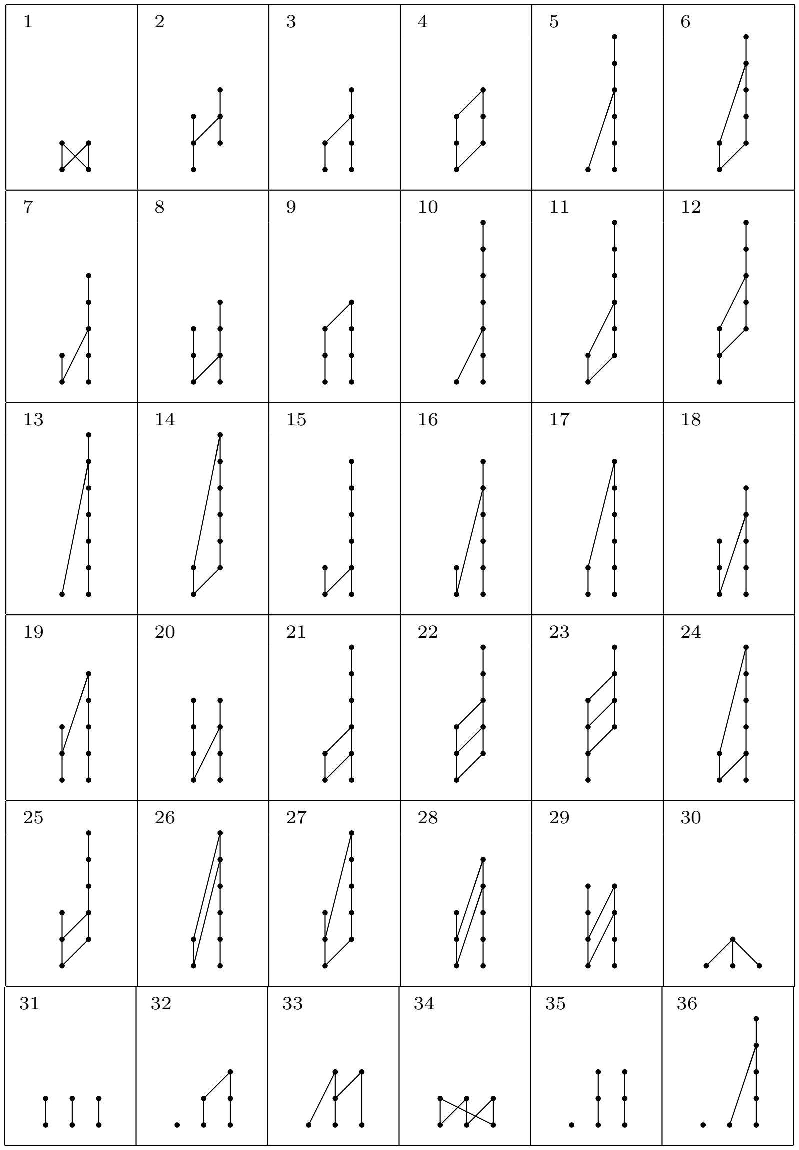
In particular, in [5] it is introduced the notion of $P$-critical poset: a poset $S$ is called $P$-*critical* if its Tits quadratic form is not positive, but that of any proper subset of $S$ is positive. If one gives the similar definition for a quiver. then the set of $P$-critical quivers coincides with the set of extended Dynkin diagrams. So the $P$-critical posets are analogs of the extended Dynkin diagrams. All such posets are classified in [5] (see the next section).

The present paper is devoted to the investigation of combinatorial properties of $P$-critical posets.

Let $S$ be a finite poset and $S^2_< := \{(x, y) \,|\, x, y \in S, x < y\}$. If $(x, y) \in S^2_<$ and there is no $z$ satisfying $x < z < y$, then one says that $x$ and $y$ are *neighboring*. We put $n_w = n_w(S) := |S^2_<|$ and denote by $n_e = n_e(S)$ the number of pairs of neighboring elements. On the language of the Hasse diagram $H(S)$ (that represents $S$ in the plane), $n_e$ is equal to the number of all its edges and $n_w$ to the number of all its paths, up to parallelity, going bottom-up (two path is called parallel if they start and terminate at the same vertices). The ratio $k_t = k_t(S)$ of the numbers $n_w - n_e$ and $n_w$ we call *the coefficient of transitiveness of* $S$. If $n_w = 0$ (then $n_e = 0$), we assume $k_t = 0$.

The aim of this paper is to calculate $k_t$ for all $P$-critical posets.

**2. Preliminary.** Indicate the table from [5], in which are written all $P$-critical posets.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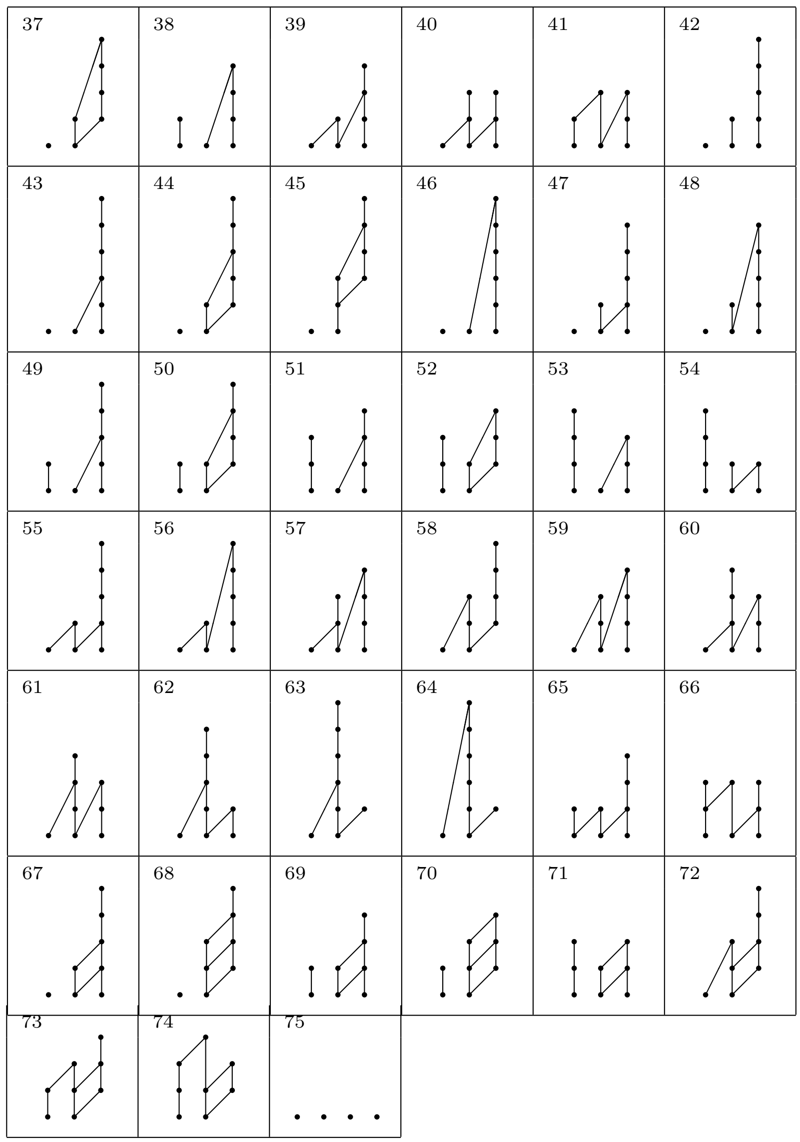


Note that the $P$-critical posets are written up to isomorphism and anti-isomorphism.

**3. Main result.** We write all the coefficients of transitiveness $k_t$ up to the second decimal place.

**Theorem 1.** *The following holds for $P$-critical posets* 1 – 75:

| $N$ | $n_e$ | $n_w$ | $k_t$ | $N$ | $n_e$ | $n_w$ | $k_t$ | $N$ | $n_e$ | $n_w$ | $k_t$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 4 | 4 | 0,00 | 26 | 8 | 19 | 0,58 | 51 | 6 | 11 | 0,45 |
| 2 | 5 | 10 | 0,50 | 27 | 8 | 19 | 0,58 | 52 | 7 | 11 | 0,36 |
| 3 | 5 | 11 | 0,55 | 28 | 8 | 16 | 0,50 | 53 | 6 | 10 | 0,40 |
| 4 | 6 | 11 | 0,45 | 29 | 8 | 15 | 0,47 | 54 | 6 | 9 | 0,33 |
| 5 | 6 | 18 | 0,67 | 30 | 3 | 3 | 0,00 | 55 | 7 | 16 | 0,56 |
| 6 | 7 | 18 | 0,61 | 31 | 3 | 3 | 0,00 | 56 | 7 | 13 | 0,46 |
| 7 | 6 | 14 | 0,57 | 32 | 4 | 6 | 0,33 | 57 | 7 | 12 | 0,42 |
| 8 | 6 | 12 | 0,50 | 33 | 5 | 7 | 0,29 | 58 | 7 | 14 | 0,50 |
| 9 | 6 | 12 | 0,50 | 34 | 6 | 6 | 0,00 | 59 | 7 | 11 | 0,36 |
| 10 | 7 | 26 | 0,73 | 35 | 4 | 6 | 0,33 | 60 | 7 | 13 | 0,46 |
| 11 | 8 | 26 | 0,69 | 36 | 5 | 12 | 0,58 | 61 | 7 | 12 | 0,42 |
| 12 | 8 | 26 | 0,69 | 37 | 6 | 12 | 0,50 | 62 | 7 | 15 | 0,53 |
| 13 | 7 | 23 | 0,70 | 38 | 5 | 8 | 0,38 | 63 | 7 | 20 | 0,65 |
| 14 | 8 | 23 | 0,65 | 39 | 6 | 10 | 0,40 | 64 | 7 | 17 | 0,59 |
| 15 | 7 | 21 | 0,67 | 40 | 6 | 10 | 0,40 | 65 | 7 | 12 | 0,42 |
| 16 | 7 | 18 | 0,61 | 41 | 6 | 8 | 0,25 | 66 | 7 | 11 | 0,36 |
| 17 | 7 | 18 | 0,61 | 42 | 5 | 11 | 0,55 | 67 | 7 | 18 | 0,61 |
| 18 | 7 | 15 | 0,53 | 43 | 6 | 19 | 0,68 | 68 | 8 | 18 | 0,56 |
| 19 | 7 | 15 | 0,53 | 44 | 7 | 19 | 0,63 | 69 | 7 | 13 | 0,46 |
| 20 | 7 | 14 | 0,50 | 45 | 7 | 19 | 0,63 | 70 | 8 | 13 | 0,38 |
| 21 | 8 | 25 | 0,68 | 46 | 6 | 16 | 0,63 | 71 | 7 | 10 | 0,30 |
| 22 | 9 | 25 | 0,64 | 47 | 6 | 15 | 0,60 | 72 | 8 | 17 | 0,53 |
| 23 | 9 | 25 | 0,64 | 48 | 6 | 12 | 0,50 | 73 | 8 | 14 | 0,43 |
| 24 | 8 | 22 | 0,64 | 49 | 6 | 14 | 0,57 | 74 | 8 | 13 | 0,38 |
| 25 | 8 | 22 | 0,64 | 50 | 7 | 14 | 0,50 | 75 | 0 | 0 | 0,00 |

The proof is carried out by direct calculations.

Here we do not analyze fully this result, and formulate only one corollary from the theorem.

Recall that an element of a poset $T$ is called *nodal*, if it is comparable with all elements of $T$. Obviously, each element of $T$ is nodal iff $T$ is a chain. It follows from the table of Section 2 that any $P$-critical poset $S$ is uniquely represented in the form $S = S_0^- 0 \cup S_1 \cup S_0^+$ where $S_0^-, S_0^+$ are chains (maybe empty), $S_1$ does not contain nodal elements and $S_0^- < S_1 < S_0^+$ ($X < Y$ means that $x < y$ for any $x \in X$, $y \in Y$). Then $S_0 = S_0^- \cup S_0^+$ is the set of all nodal elements of $S$.

**Corollary 1.** *Let $S$ be a $P$-critical poset. Then the following conditions are equivalent*:

a) $k_t(S) \geq k_t(T)$ for any $P$-critical poset $T$;

b) $|S_0| \geq |T_0|$ for any $P$-critical poset $T$, and $S_0^-$ or $S_0^+$ is empty.

## References

[1] *Gabriel P.* Unzerlegbare Darstellungen // Manuscripta Math. — 1972. — **6**. — pp. 71 – 103.

[2] *Drozd Yu. A.* Coxeter transformations and representations of partially ordered sets // Funkc. Anal. i Priložen. — 1974. — **8**, No. 3. — pp. 34 – 42 (in Russian).

[3] *Nazarova L. A., Roiter A. V.* Representations of posets // Zap. Nauch. Semin. LOMI. — 1972. — **28**. — pp. 5 – 31 (in Russian).

[4] *Bondarenko V. M., Styopochkina M. V.* On posets of width two with positive Tits form // Algebra and Discr. Math. — 2005. — No. 2. — pp. 11 – 22.

[5] *Bondarenko V. M., Stepochkina M. V.* (Min, max)-equivalence of partially ordered sets and the Tits quadratic form // Zb. Pr. Inst. Mat. NAN Ukr./Problems of Analysis and Algebra/. — K.: Institute of Mathematics of NAN of Ukraine, 2005. — **2**, No. 3. — pp. 18 – 58 (in Russian).

[6] *Bondarenko V. M., Stepochkina M. V.* On finite posets of inj-finite type and their Tits forms // Algebra and Discr. Math. — 2006. — No. 2. — pp. 17 – 21.

[7] *Bondarenko V. M., Stepochkina M. V.* On the serial posets with positive-definite quadratic Tits form // Nelin. Kolyv. — 2006. — **9**, No. 3. — pp. 320 – 325 (in Ukrainian).