DOI 10.1007/s10958-018-3687-2
Journal of Mathematical Sciences, Vol. 229, No. 4, March, 2018

WEAKLY PERTURBED FREDHOLM INTEGRAL EQUATIONS WITH DEGENERATE KERNELS IN BANACH SPACES

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We consider weakly perturbed Fredholm equations with degenerate kernels in Banach spaces and establish conditions for $\varepsilon = 0$ to be a bifurcation point for the solutions of weakly perturbed operator equations in Banach spaces. A convergent iterative scheme for finding solutions in the form of series $\sum_{i=-1}^{+\infty} \varepsilon^i z_i(t)$ in powers of ε is proposed.

The investigation of the conditions of solvability and the construction of solutions of weakly perturbed Fredholm integral equations with degenerate kernels in Banach spaces continue the development of the methods of perturbation theory and, in particular, of the Lyapunov–Poincaré [1] and Vishik–Lyusternik [2] methods of small parameter.

These methods were successfully used for the construction of solutions of weakly perturbed boundary-value problems for systems of ordinary differential equations and functional-differential equations with Noetherian linear parts [3–5] in the Euclidean spaces. The approach to the investigation of differential systems in Banach spaces proposed in [6] was applied by Boichuk and Panasenko in [7] for the investigation of weakly perturbed boundary-value problems for systems of ordinary differential equations in Banach spaces.

It is worth noting that a differential system for the linear generating boundary-value problem ($\varepsilon = 0$) has solutions for any right-hand side, i.e., according to the S. Krein classification [8], it is everywhere solvable. In [9, 10], Boichuk, Shegda, and Holovats'ka studied weakly perturbed singular differential and integrodifferential equations that are not everywhere solvable in finite-dimensional spaces.

Hence, the problem of investigation of the conditions required for the appearance of solutions of weakly perturbed Fredholm integral equations with degenerate kernels in Banach spaces seems to be quite urgent.

Statement of the Problem

In a Banach space $\mathbf{B}$, we consider a weakly perturbed Fredholm equation

$$(Lz)(t) := z(t) - M(t)\int_a^b N(s)z(s)ds = f(t) + \varepsilon \int_a^b K(t,s)z(s)ds, \tag{1}$$

where the operator functions $M(t)$ and $N(t)$ act from the real Banach space $\mathbf{B}$ into the same space and are strongly continuous [6] with the norms

$$|||M||| = \sup_{t\in\mathcal{I}} \|M(t)\|_{\mathbf{B}} = M_0 < \infty$$

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Translated from Neliniini Kolyvannya, Vol. 20, No. 1, pp. 85–97, January–March, 2017. Original article submitted October 17, 2016.

and

$$|||N||| = \sup_{t \in \mathcal{I}} \|N(t)\|_{\mathbf{B}} = N_0 < \infty,$$

respectively, the operator function $K(t, s)$ is defined in the square $\mathcal{I} \times \mathcal{I}$, acts from the Banach space $\mathbf{B}$ into $\mathbf{B}$ with respect to each variable, and is strongly continuous in t and s with the norm

$$|||K||| = \sup_{t,s \in \mathcal{I}} \|K(t, s)\|_{\mathbf{B}} = K_0 < \infty,$$

the vector function $f(t)$ acts from the segment $\mathcal{I}$ into the Banach space $\mathbf{B}$:

$$f(t) \in \mathbf{C}(\mathcal{I}, \mathbf{B}) := \Big\{ f(\cdot)\colon \mathcal{I} \to \mathbf{B},\, |||f||| = \sup_{t \in \mathcal{I}} \|f(t)\| \Big\},$$

$\mathbf{C}(\mathcal{I}, \mathbf{B})$ is a Banach space of vector functions continuous on $\mathcal{I}$ and taking values in $\mathbf{B}$, and $\varepsilon << 1$ is a small parameter.

Assume that the generating equation

$$(Lz)(t) := z(t) - M(t) \int\limits_a^b N(s) z(s) ds = f(t) \tag{2}$$

obtained from (1) for $\varepsilon = 0$ does not have solutions for any inhomogeneity $f(t) \in \mathbf{C}(\mathcal{I}, \mathbf{B})$.

In the present paper, by using the theory of generalized inversion of the operators [4, 5] and, in particular, of the generalized inversion of Fredholm integral operators with degenerate kernels in Banach spaces [11], as well as the theorems on solvability of equations with generalized invertible operators L [12, 13] , we consider the problems of finding the conditions for the appearance of solutions of Eq. (2) perturbed by a small linear term $\varepsilon \int_a^b K(t, s) z(s) ds$ and the construction of the general solution of Eq. (1).

First, we present some known results necessary in what follows.

Auxiliary Information

Let $z(t) \in \mathbf{C}(\mathcal{I}, \mathbf{B})$ be a vector function acting from the segment $\mathcal{I} = [a, b]$ into the Banach space $\mathbf{B}$.

In the Banach space $\mathbf{B}$, we consider the linear Fredholm integral equation with degenerate kernel (2).

Denote

$$D = I_{\mathbf{B}} - A, \quad A = \int\limits_a^b N(s) M(s)\, ds, \quad D\colon \mathbf{B} \to \mathbf{B}.$$

In [11], it is shown that if D is a bounded generalized invertible operator, then the integral operator L is generalized invertible.

To prove the generalized invertibility of the integral operator, we construct the projectors

$$(\mathcal{P}_{N(L)} z)(t) = M(t) \mathcal{P}_{N(D)} \int\limits_a^b N(s) z(s) ds, \quad \mathcal{P}_{N(L)}\colon \mathbf{C}(\mathcal{I}, \mathbf{B}) \to N(L),$$

$$(\mathcal{P}_{Y_L} f)(t) = M(t)\mathcal{P}_{Y_D} \int_a^b N(s)f(s)ds, \quad \mathcal{P}_{Y_L}\colon \mathbf{C}(\mathcal{I}, \mathbf{B}) \to Y_L$$

and prove that they are bounded. Here, $\mathcal{P}_{N(D)}$ and $\mathcal{P}_{Y_D}$ are bounded projectors onto the null space $N(D)$ and onto the subspace Y_D of the operator D, respectively, [14], which split the Banach space $\mathbf{B}$ into the direct topological sums of closed subspaces

$$\mathbf{B} = N(D) \oplus X_D, \quad \mathbf{B} = Y_D \oplus R(D).$$

In what follows, by $\mathbf{GI}(\mathbf{C}(\mathcal{I}, \mathbf{B}), \mathbf{C}(\mathcal{I}, \mathbf{B}))$ we denote the class of linear bounded generalized invertible operators acting from the Banach space $\mathbf{C}(\mathcal{I}, \mathbf{B})$ into the Banach space $\mathbf{C}(\mathcal{I}, \mathbf{B})$. It is clear that the operators from $\mathbf{GI}(\mathbf{C}(\mathcal{I}, \mathbf{B}), \mathbf{C}(\mathcal{I}, \mathbf{B}))$ are normally solvable.

Theorem 1 [11]. *Suppose that D belongs to $\mathbf{GI}(\mathbf{B}, \mathbf{B})$. Then the operator*

$$(L^- f)(t) = f(t) + M(t)D^- \int_a^b N(s)f(s)ds \tag{3}$$

is a bounded generalized inverse operator for the integral operator L, where D^- is a bounded generalized inverse operator for the operator D.

Theorem 2 [11]. *Suppose that D belongs to $\mathbf{GI}(\mathbf{B}, \mathbf{B})$. Then the homogeneous integral equation for (2) has the family of solutions*

$$z(t) = M(t)\mathcal{P}_{N(D)}c,$$

where c is an arbitrary element of the Banach space $\mathbf{B}$.

Under the condition

$$\mathcal{P}_{Y_D} \int_a^b N(s)f(s)ds = 0$$

and only under this condition, the inhomogeneous integral equation (2) has the family of solutions

$$z(t) = M(t)\mathcal{P}_{N(D)}c + (L^- f)(t),$$

where L^- is the bounded generalized inverse operator (3) for the operator L.

Main Result

To solve the posed problem, we use the Vishik–Lyusternik method [2] and determine the conditions for the appearance of solutions of the integral equation (1) in the form of power series in the small parameter ε containing the negative powers of ε. We seek the solution of Eq. (1) in the form of a series

$$z(t, \varepsilon) = \sum_{i=-1}^{+\infty} \varepsilon^i z_i(t). \tag{4}$$

We substitute series (4) in the integral equation (1) and equate the coefficients of the same powers of ε. For ε^{-1}, we arrive at the following homogeneous integral equation:

$$z_{-1}(t) - M(t)\int_a^b N(s)z_{-1}(s)ds = 0 \tag{5}$$

for $z_{-1}(t)$.

By Theorem 2, the homogeneous equation (5) possesses the solution

$$z_{-1}(t, c_{-1}) = M(t)\mathcal{P}_{N(D)}c_{-1}, \tag{6}$$

where $c_{-1} \in \mathbf{B}$ is an arbitrary element determined in what follows.

Equating the coefficients of ε^0, we arrive at the following inhomogeneous integral equation:

$$z_0(t) - M(t)\int_a^b N(s)z_0(s)ds = f(t) + \int_a^b K(t,s)z_{-1}(s)ds \tag{7}$$

for the coefficient $z_0(t)$ of series (4).

By Theorem 2, the linear homogeneous integral equation (7) possesses solutions if and only if

$$\mathcal{P}_{Y_D}\int_a^b N(s)\left[f(s) + \int_a^b K(s,\tau)z_{-1}(\tau)d\tau\right]ds = 0.$$

Substituting $z_{-1}(t, c_{-1})$ from (6) in the last equation, we get

$$\mathcal{P}_{Y_D}\int_a^b N(s)\left[f(s) + \int_a^b K(s,\tau)M(\tau)\mathcal{P}_{N(D)}c_{-1}d\tau\right]ds = 0. \tag{8}$$

Denote

$$B_0 = \mathcal{P}_{Y_D}\int_a^b N(s)\int_a^b K(s,\tau)M(\tau)\mathcal{P}_{N(D)}d\tau ds. \tag{9}$$

This enables us to derive the following operator equation for the element $c_{-1} \in \mathbf{B}$ from (8):

$$B_0 c_{-1} = -\mathcal{P}_{Y_D}\int_a^b N(s)f(s)ds. \tag{10}$$

Let the operator $B_0 \in \mathbf{GI}(\mathbf{B}, Y_D)$ be generalized invertible. Then it is normally solvable and there exist bounded projectors

$$\mathcal{P}_{N(B_0)}\colon \mathbf{B} \to N(B_0) \quad \text{and} \quad \mathcal{P}_{Y_{B_0}}\colon \mathbf{B} \to Y_{B_0}$$

and a bounded generalized inverse operator $B_0^-\colon \mathbf{B} \to \mathbf{B}$ for the operator B_0.

Equation (10) may be [8]: uniquely solvable ($\mathcal{P}_{N(B_0)} \equiv 0$), everywhere solvable ($\mathcal{P}_{Y_{B_0}} \equiv 0$), and not uniquely and not everywhere solvable ($\mathcal{P}_{N(B_0)} \neq 0$ and $\mathcal{P}_{Y_{B_0}} \neq 0$).

We now consider the most general case where Eq. (10) is not uniquely and not everywhere solvable. By Theorem 2, in view of the generalized invertibility of the operator B_0, Eq. (10) is solvable if and only if its right-hand side satisfies the condition

$$\mathcal{P}_{Y_{B_0}} \mathcal{P}_{Y_D} \int_a^b N(s) f(s) ds = 0.$$

This condition is satisfied whenever

$$\mathcal{P}_{Y_{B_0}} \mathcal{P}_{Y_D} = 0, \tag{11}$$

and the operator equation (10) possesses a family of solutions

$$c_{-1} = \mathcal{P}_{N(B_0)} c - B_0^- \mathcal{P}_{Y_D} \int_a^b N(s) f(s)\, ds,$$

where c is an arbitrary element of the Banach space $\mathbf{B}$.

Substituting c_{-1} in (6), we get the general solution of the homogeneous integral equation (5)

$$z_{-1}(t) = M(t) \mathcal{P}_{N(D)} \mathcal{P}_{N(B_0)} c - M(t) \mathcal{P}_{N(D)} B_0^- \mathcal{P}_{Y_D} \int_a^b N(s) f(s) ds.$$

Denoting

$$\widetilde{B}_0^- = -\mathcal{P}_{N(D)} B_0^- \mathcal{P}_{Y_D}, \tag{12}$$

we finally obtain

$$z_{-1}(t) = M(t) \mathcal{P}_{N(D)} \mathcal{P}_{N(B_0)} c + M(t) \widetilde{B}_0^- \int_a^b N(s) f(s)\, ds.$$

If relation (11) is true, then condition (8) is satisfied and, by Theorem 2, the inhomogeneous integral equation (7) possesses a family of solutions

$$z_0(t, c_0) = M(t) \mathcal{P}_{N(D)} c_0 + \bar{z}_0(t), \tag{13}$$

where c_0 is an arbitrary element determined in the next step and

$$\bar{z}_0(t) = L^- \left[f(\cdot) + \int_a^b K(\cdot, s) z_{-1}(s) ds \right](t)$$

$$= L^{-}\left(f(\cdot) + \int_a^b K(\cdot, s)M(s)\mathcal{P}_{N(D)}\mathcal{P}_{N(B_0)}ds\ c\right.$$

$$\left. + \int_a^b K(\cdot, s)M(s)\widetilde{B}_0^{-}\int_a^b N(\tau)f(\tau)d\tau ds\right)(t) = H_{-1}(t)\mathcal{P}_{N(B_0)}c + \widetilde{F}_{-1}(t).$$

Here,

$$H_{-1}(t) = \left(L^{-}(\widetilde{K}M)(\cdot)\mathcal{P}_{N(D)}\right)(t),$$

$$\widetilde{F}_{-1}(t) = L^{-}\left[f(\cdot) + (\widetilde{K}M(\cdot)\widetilde{B}_0^{-}\int_a^b N(\tau)f(\tau)d\tau\right](t),$$

and the operator $\widetilde{K}$ acts upon an operator function $M(t)$ by the rule

$$(\widetilde{K}M)(t) = \int_a^b K(t, s)M(s)ds.$$

It follows from relation (3) that the action of the operator L^{-} upon the vector function

$$f(t) + \widetilde{K}M(t)\widetilde{B}_0^{-}\int_a^b N(s)f(s)ds$$

is described by the formula

$$L^{-}\left[f(\cdot) + \widetilde{K}M(\cdot)\widetilde{B}_0^{-}\int_a^b N(s)f(s)ds\right](t) = f(t) + \widetilde{K}M(t)\widetilde{B}_0^{-}\int_a^b N(s)f(s)ds$$

$$+ M(t)D^{-}\int_a^b N(s)\left[f(s) + (\widetilde{K}M)(s)\widetilde{B}_0^{-}\int_a^b N(\tau)f(\tau)d\tau\right]ds.$$

For ε^1, we arrive at the following equation for the coefficient $z_1(t)$:

$$z_1(t) - M(t)\int_a^b N(s)z_1(s)ds = \int_a^b K(t, s)z_0(s)ds. \tag{14}$$

In view of the criterion of solvability for Eq. (14)

$$\mathcal{P}_{Y_D} \int_a^b N(s) \int_a^b K(s,\tau) z_0(\tau) d\tau ds = 0,$$

by virtue of (13), we get

$$\mathcal{P}_{Y_D} \int_a^b N(s) \int_a^b K(s,\tau) \left[M(\tau) \mathcal{P}_{N(D)} c_0 + \bar{z}_0(\tau) \right] d\tau ds = 0.$$

By using notation (9), we derive the following operator equation for the element $c_0 \in \mathbf{B}$ from the last equation:

$$B_0 c_0 = -\mathcal{P}_{Y_D} \int_a^b N(s) \int_a^b K(s,\tau) \bar{z}_0(\tau) d\tau ds = -\mathcal{P}_{Y_D} \int_a^b N(s) (\widetilde{K} \bar{z}_0)(s) ds, \tag{15}$$

where

$$(\widetilde{K} \bar{z}_0)(s) = \int_a^b K(s,\tau) \bar{z}_0(\tau) d\tau.$$

Under condition (11), the operator equation (15) has the following family of solutions:

$$\begin{aligned} c_0 &= \mathcal{P}_{N(B_0)} c - B_0^- \mathcal{P}_{Y_D} \int_a^b N(s) (\widetilde{K} \bar{z}_0)(s)\, ds \\ &= \mathcal{P}_{N(B_0)} c - B_0^- \mathcal{P}_{Y_D} \int_a^b N(s) \left(\widetilde{K} \left[H_{-1}(s) \mathcal{P}_{N(B_0)} c + \widetilde{F}_{-1}(s) \right] \right)(s)\, ds \\ &= D_0 \mathcal{P}_{N(B_0)} c + \bar{c}_0, \end{aligned} \tag{16}$$

where

$$D_0 = I_{\mathbf{B}} - B_0^- \mathcal{P}_{Y_D} \int_a^b N(s) (\widetilde{K} H_{-1})(s) ds,$$

$$\bar{c}_0 = -B_0^- \mathcal{P}_{Y_D} \int_a^b N(s) (\widetilde{K} \widetilde{F}_{-1})(s) ds,$$

and $I_{\mathbf{B}}$ is the identity operator in the Banach space $\mathbf{B}$.

Substituting (16) in (13) and using (12), we find

$$z_0(t) = M(t)\mathcal{P}_{N(D)}\left[D_0\mathcal{P}_{N(B_0)}c + \bar{c}_0\right] + \bar{z}_0(t)$$

$$= M(t)\mathcal{P}_{N(D)}D_0\mathcal{P}_{N(B_0)}c + M(t)\mathcal{P}_{N(D)}\bar{c}_0 + H_{-1}(t)\mathcal{P}_{N(B_0)}c + \widetilde{F}_{-1}(t)$$

$$= X_0(t)\mathcal{P}_{N(B_0)}c + \bar{z}_0(t),$$

where

$$X_0(t) = H_{-1}(t) + M(t)\mathcal{P}_{N(D)}D_0 = H_{-1}(t)$$

$$+ M(t)\mathcal{P}_{N(D)}\left[I - B_0^-\mathcal{P}_{Y_D}\int_a^b N(s)(\widetilde{K}H_{-1})(s)ds\right]$$

$$= M(t)\mathcal{P}_{N(D)} + \left[I * + M(t)\widetilde{B}_0^-\int_a^b N(s)(\widetilde{K}*)ds\right]H_{-1}(t),$$

$$\bar{z}_0(t) = M(t)\mathcal{P}_{N(D)}\bar{c}_0 + \widetilde{F}_{-1}(t) = \left[I * + M(t)\widetilde{B}_0^-\int_a^b N(s)\left(\widetilde{K}*\right)\,ds\right]\widetilde{F}_{-1}(t).$$

Here, I is the identity operator in the Banach space $\mathbf{C}(\mathcal{I}, \mathbf{B})$.

If Eq. (15) is true, then the integral equation (14) possesses a family of solutions

$$z_1(t, c_1) = M(t)\mathcal{P}_{N(D)}c_1 + \bar{z}_1(t), \tag{17}$$

where $c_1 \in \mathbf{B}$ is an arbitrary element determined in the next step of the iterative process,

$$\bar{z}_1(t) = L^-\left(\widetilde{K}z_0\right)(t) = L^-\widetilde{K}\left(\left[X_0(t)\mathcal{P}_{N(B_0)}c + \bar{z}_0(t)\right]\right)(t) = H_0(t)\mathcal{P}_{N(B_0)}c + \widetilde{F}_0(t),$$

$$H_0(t) = L^-(\widetilde{K}X_0)(t),$$

$$\widetilde{F}_0(t) = L^-\left(\widetilde{K}\left[I * + M(\cdot)\widetilde{B}_0^-\int_a^b N(s)\left(\widetilde{K}*\right)ds\right]\widetilde{F}_{-1}(\cdot)\right)(t).$$

By induction, we derive the following equations for the coefficients $z_i(t)$ of ε^i in series (4):

$$z_i(t) - M(t)\int_a^b N(s)z_i(s)\,ds = \int_a^b K(t,s)z_{i-1}(s)ds, \quad i = 1, 2, 3, \ldots. \tag{18}$$

By using the solvability criteria for Eqs. (18)

$$\mathcal{P}_{Y_D}\int_a^b N(s)\int_a^b K(s,\tau)z_{i-1}(\tau)\,d\tau\,ds = 0,$$

we arrive at the following operator equations for c_i:

$$B_0c_{i-1} = -\mathcal{P}_{Y_D}\int_a^b N(s)\int_a^b K(s,\tau)\bar{z}_{i-1}(\tau)\,d\tau\,ds = -\mathcal{P}_{Y_D}\int_a^b N(s)(\widetilde{K}\bar{z}_{i-1})(s)\,ds. \tag{19}$$

Under condition (11), the operator equations (19) have the families of solutions

$$\begin{aligned} c_{i-1} &= \mathcal{P}_{N(B_0)}c - B_0^{-}\mathcal{P}_{Y_D}\int_a^b N(s)(\widetilde{K}\bar{z}_{i-1})(s)\,ds \\ &= \mathcal{P}_{N(B_0)}c - B_0^{-}\mathcal{P}_{Y_D}\int_a^b N(s)\left(\widetilde{K}\left[H_{i-2}(s)\mathcal{P}_{N(B_0)}c + \widetilde{F}_{i-2}(t)\right]\right)(s)\,ds \\ &= D_{i-1}\mathcal{P}_{N(B_0)}c + \bar{c}_{i-1}, \end{aligned}$$

where

$$D_{i-1} = I_{\mathbf{B}} - B_0^{-}\mathcal{P}_{Y_D}\int_a^b N(s)(\widetilde{K}H_{i-2})(s)ds,$$

$$\bar{c}_{i-1} = -B_0^{-}\mathcal{P}_{Y_D}\int_a^b N(s)(\widetilde{K}\widetilde{F}_{i-2})(s)\,ds.$$

Under condition (11) and, hence, conditions (19), the integral equations (18) possess the families of solutions

$$\begin{aligned} z_i(t,c_i) &= M(t)\mathcal{P}_{N(D)}c_i + \bar{z}_i(t) \\ &= M(t)\mathcal{P}_{N(D)}[D_i\mathcal{P}_{N(B_0)}c + \bar{c}_i] + \bar{z}_i(t) \\ &= M(t)\mathcal{P}_{N(D)}D_i\mathcal{P}_{N(B_0)}c + M(t)\mathcal{P}_{N(D)}\bar{c}_i + H_{i-1}(t)\mathcal{P}_{N(B_0)}c + \widetilde{F}_i(t) \\ &= X_i(t)\mathcal{P}_{N(B_0)}c + \bar{z}_i(t), \end{aligned}$$

where

$$X_i(t) = H_{i-1}(t) + M(t)\mathcal{P}_{N(D)}D_i = H_{i-1}(t)$$

$$+ M(t)\mathcal{P}_{N(D)}\left[I - B_0^{-}\mathcal{P}_{Y_D}\int\limits_a^b N(s)(\widetilde{K}H_{i-1})(s)ds\right]$$

$$= M(t)\mathcal{P}_{N(D)} + \left[I * + M(t)\widetilde{B}_0^{-}\int\limits_a^b N(s)(\widetilde{K}*)ds\right]H_{i-1}(t),$$

$$\bar{z}_i(t) = M(t)\mathcal{P}_{N(D)}\bar{c}_i + \widetilde{F}_{i-1}(t) = \left[I * + M(t)\widetilde{B}_0^{-}\int\limits_a^b N(s)\left(\widetilde{K}*\right)ds\right]\widetilde{F}_{i-1}(t).$$

The presented reasoning enables us to propose the following iterative algorithm for the construction of a family of solutions of the integral equation (1):

$$z_i(t, c_i) = M(t)\mathcal{P}_{N(D)}\mathcal{P}_{N(B_0)}c + \widetilde{X}_i(t)\mathcal{P}_{N(B_0)}c + \bar{z}_i(t),$$

$$\widetilde{X}_i(t) = \begin{cases} 0 & \text{for} \quad i = -1, \\ \left[I * + M(t)\widetilde{B}_0^{-}\int_a^b N(s)\left(\widetilde{K}*\right)ds\right]H_{i-1}(t) & \text{for} \quad i = \overline{0,\infty}, \end{cases}$$

$$H_{i-1}(t) = \begin{cases} \left(L^{-}(\widetilde{K}M(\cdot)\mathcal{P}_{N(D)})\right)(t) & \text{for} \quad i = 0, \\ \left(L^{-}(\widetilde{K}X_{i-1}(\cdot))\right)(t) & \text{for} \quad i = \overline{1,\infty}, \end{cases} \tag{20}$$

$$\bar{z}_i(t) = \begin{cases} M(t)\widetilde{B}_0^{-}\int_a^b N(s)f(s)ds & \text{for} \quad i = -1, \\ \left[I * + M(t)\widetilde{B}_0^{-}\int_a^b N(s)\left(\widetilde{K}*\right)ds\right]\widetilde{F}_{i-1}(t) & \text{for} \quad i = \overline{0,\infty}, \end{cases}$$

$$\widetilde{F}_{i-1}(t) = \begin{cases} \left(L^{-}\left[I * + (\widetilde{K}M)(\cdot)\widetilde{B}_0^{-}\int_a^b N(s) * ds\right]f(\cdot)\right)(t) & \text{for} \quad i = 0, \\ L^{-}\left(\widetilde{K}\left[I * + M(\cdot)\widetilde{B}_0^{-}\int_a^b N(s)\left(\widetilde{K}*\right)ds\right]\widetilde{F}_{i-2}(\cdot)\right)(t) & \text{for} \quad i = \overline{1,\infty}. \end{cases}$$

Thus, under condition (11), the weakly perturbed operator equation (1) has a family of solutions in the form of a series

$$z(t,\varepsilon) = \sum_{i=-1}^{+\infty} \varepsilon^i z_i(t)$$

$$= \sum_{i=-1}^{+\infty} \varepsilon^i M(t)\mathcal{P}_{N(D)}\mathcal{P}_{N(B_0)}c + \sum_{i=0}^{+\infty} \varepsilon^i \widetilde{X}_i(t)\mathcal{P}_{N(B_0)}c + \sum_{i=-1}^{+\infty} \varepsilon^i \bar{z}_i(t). \tag{21}$$

By using the notation

$$|||M||| = \sup_{t\in\mathcal{I}}\|M(t)\|_{\mathbf{B}} = M_0 < \infty, \quad |||N||| = \sup_{t\in\mathcal{I}}\|N(t)\|_{\mathbf{B}} = N_0 < \infty,$$

$$|||\widetilde{B}_0^-|||_{\mathbf{B}} = \tilde{b}_0 < \infty, \quad |||L^-|||_{\mathbf{C}(\mathcal{I},\mathbf{B})} = l < \infty,$$

$$|||\widetilde{K}|||_{\mathbf{C}(\mathcal{I},\mathbf{B})} = \tilde{k} < \infty, \quad |||f(t)|||_{\mathbf{C}(\mathcal{I},\mathbf{B})} = f < \infty,$$

$$|||\mathcal{P}_{N(D)}|||_{\mathbf{B}} = p < \infty, \quad |||\mathcal{P}_{N(B_0)}|||_{\mathbf{B}} = \tilde{p} < \infty,$$

we establish the uniform convergence of series (21) for fixed $\varepsilon \in (0, \varepsilon_*]$.

It is clear that, by virtue of boundedness of the operators $M(t)$, $\mathcal{P}_{N(D)}$, and $\mathcal{P}_{N(B_0)}$, the series

$$\sum_{i=-1}^{+\infty} \varepsilon^i M(t)\mathcal{P}_{N(D)}\mathcal{P}_{N(B_0)}c = M(t)\mathcal{P}_{N(D)}\mathcal{P}_{N(B_0)}c \sum_{i=-1}^{+\infty} \varepsilon^i$$

converges for $\varepsilon < 1$.

Further, we prove convergence of the series

$$\sum_{i=0}^{+\infty} \varepsilon^i \widetilde{X}_i \mathcal{P}_{N(B_0)}c. \tag{22}$$

For $i = 0$, we get

$$|||\widetilde{X}_0(t)|||_{\mathbf{C}(\mathcal{I},\mathbf{B})} \leq lkM_0 p(1 + M_0 N_0 \tilde{b}_0 k).$$

Similarly, for $i = 1$, we find

$$|||\widetilde{X}_1|||_{\mathbf{C}(\mathcal{I},\mathbf{B})} \leq lk(1 + M_0 N_0 \tilde{b}_0 k)|||\widetilde{X}_0|||_{\mathbf{C}(\mathcal{I},\mathbf{B})}.$$

Continuing this process, we obtain the following estimates for the operators $\widetilde{X}_i$:

$$|||\widetilde{X}_i|||_{\mathbf{l}_\infty(\mathcal{I},\mathbf{B}_1)} \leq [lk(1 + M_0 N_0 \tilde{b}_0 k)]^i |||\widetilde{X}_0|||_{\mathbf{C}(\mathcal{I},\mathbf{B})}.$$

Thus, for any $t \in \mathcal{I}$, we can write

$$\sum_{i=0}^{+\infty} \varepsilon^i \widetilde{X}_i(t) \mathcal{P}_{N(B_0)} c \leq \sum_{i=0}^{+\infty} \varepsilon^i K_1^i |||\widetilde{X}_0(t)|||_{\mathbf{C}(\mathcal{I},\mathbf{B})} |||\mathcal{P}_{N(B_0)}|||_{\mathbf{B}} |||c|||_{\mathbf{B}},$$

where

$$K_1 = lk(1 + M_0 N_0 \tilde{b}_0 k).$$

Hence, for fixed $\varepsilon \in (0, \varepsilon_*]$, where $\varepsilon_* < K_1^{-1}$, series (22) is uniformly convergent.

Similarly, we prove convergence of the series

$$\sum_{i=-1}^{+\infty} \varepsilon^i \bar{z}_i(t).$$

Thus, we get the following estimate for the coefficients $\bar{z}_i(t)$, $i = \overline{1, \infty}$:

$$|||\bar{z}_i(t)|||_{\mathbf{C}(\mathcal{I},\mathbf{B})} \leq k^i l^{i+1} (1 + M_0 N_0 \tilde{b}_0 k)^{i+2} |||f(t)|||_{\mathbf{C}(\mathcal{I},\mathbf{B})}.$$

Hence, for any $t \in \mathcal{I}$, we find

$$\sum_{i=-1}^{+\infty} \varepsilon^i \bar{z}_i(t) \leq \varepsilon^{-1} M_0 N_0 \tilde{b}_0 |||f(t)|||_{\mathbf{C}(\mathcal{I},\mathbf{B})}$$

$$+ \sum_{i=0}^{+\infty} \varepsilon^i k^i l^{i+1} (1 + M_0 N_0 \tilde{b}_0 k)^{i+2} |||f(t)|||_{\mathbf{C}(\mathcal{I},\mathbf{B})} = \varepsilon^{-1} M_0 N_0 \tilde{b}_0 |||f(t)|||_{\mathbf{C}(\mathcal{I},\mathbf{B})}$$

$$+ \sum_{i=0}^{+\infty} \varepsilon^i K_1^i l (1 + M_0 N_0 \tilde{b}_0 k)^2 |||f(t)|||_{\mathbf{C}(\mathcal{I},\mathbf{B})}$$

and, for fixed $\varepsilon \in (0, \varepsilon_*]$, where $\varepsilon_* < K_1^{-1}$, the series

$$\sum_{i=-1}^{+\infty} \varepsilon^i \bar{z}_i(t)$$

uniformly converges.

Let

$$\varepsilon_* < \min(1, K_1^{-1}).$$

Then, for $\varepsilon \in (0, \varepsilon_*]$, series (21) is uniformly convergent.

Theorem 3. *Suppose that the operator D belongs to $\mathbf{GI}(\mathbf{B}, \mathbf{B})$ and, for any inhomogeneity $f(t) \in \mathbf{C}(\mathcal{I}, \mathbf{B})$, the generating equation (2) has no solutions.*

If the operator B_0 belongs to $\mathbf{GI}(\mathbf{B}, \mathbf{B})$ and

$$\mathcal{P}_{Y_{B_0}} \mathcal{P}_{Y_D} = 0,$$

then, for any inhomogeneity

$$f(t) \in \mathbf{C}(\mathcal{I}, \mathbf{B}),$$

the weakly perturbed equation (1) possesses a family of solutions in the form of an absolutely convergent series

$$x(t, \varepsilon) = \sum_{i=-1}^{+\infty} \varepsilon^i z_i(t),$$

for any fixed $\varepsilon \in (0, \varepsilon_*]$ and the coefficients of the series are determined according to the iterative algorithm (20).

Remark 1. If

$$\mathcal{P}_{N(B_0)} = 0,$$

then, in each step of the iterative process, the operator equations (10), (15), etc., are n-normal and uniquely solvable [8]. Hence, under the condition

$$\mathcal{P}_{Y_{B_0}} \mathcal{P}_{Y_D} = 0,$$

Eq. (1) possesses a unique solution in the form of series (4) whose coefficients are determined by the iterative algorithm (20) in which $\mathcal{P}_{N(B_0)} = 0$ and the generalized inverse operator B_0^- is equal to the left inverse operator $(B_0)_l^{-1}$ [15].

Remark 2. If

$$\mathcal{P}_{Y_{B_0}} = 0,$$

then, in each step of the iterative process, the operator equations (10), (15), etc., are d-normal and everywhere solvable [8]. Then condition (11) is always satisfied and, for any $f(t)$, Eq. (1) has a family of solutions in the form of series (4) whose coefficients are determined by the iterative algorithm (20) with the generalized inverse operator B_0^- equal to the right inverse operator $(B_0)_r^{-1}$ [15].

REFERENCES

1. A. M. Lyapunov, *General Problem of Stability of Motion* [in Russian], Gostekhizdat, Moscow (1950).
2. M. I. Vishik and L. A. Lyusternik, “Solution of some problems of perturbations in the case of matrices and self-adjoint and nonself-adjoint differential equations,” *Usp. Mat. Nauk,* **15**, Issue 3, 3–80 (1960).
3. A. A. Boichuk, *Constructive Methods for the Analysis of Boundary-Value Problems* [in Russian], Naukova Dumka, Kiev (1990).
4. A. A. Boichuk, V. F. Zhuravlev, and A. M. Samoilenko, *Generalized Inverse Operators and Fredholm Boundary-Value Problems* [in Russian], Institute of Mathematics, Ukrainian National Academy of Sciences, Kiev (1995).
5. A. A. Boichuk and A. M. Samoilenko, *Generalized Inverse Operators and Fredholm Boundary-Value Problems,* VSP, Utrecht–Boston (2004).

6. Yu. L. Daletskii and M. G. Krein, *Stability of Solutions of Differential Equations in Banach Spaces* [in Russian], Nauka, Moscow (1970).
7. O. A. Boichuk and E. V. Panasenko, "Weakly perturbed boundary-value problems for differential equations in a Banach space," *Nelin. Kolyv.,* **13**, No. 3, 291–304 (2010); ***English translation:*** *Nonlin. Oscillat.,* **13**, No. 3, 311–324 (2011).
8. S. G. Krein, *Linear Equations in Banach Spaces* [in Russian], Nauka, Moscow (1971).
9. A. A. Boichuk and L. M. Shegda, "Bifurcation of solutions of singular Fredholm boundary value problems," *Different. Equat.,* **47**, No. 4, 453–461 (2011).
10. I. A. Holovats'ka, "Weakly perturbed systems of linear integrodifferential equations," *Nelin. Kolyv.,* **15**, No. 2, 151–164 (2012); ***English translation:*** *J. Math. Sci.,* **189**, No. 5, 735–750 (2013).
11. V. P. Zhuravl'ov, "Generalized inversion of Fredholm integral operators with degenerate kernels in Banach spaces," *Nelin. Kolyv.,* **17**, No. 3, 351–364 (2014); ***English translation:*** *J. Math. Sci., J. Math. Sci..,* **212**, No. 3, 275–289 (2015).
12. A. A. Boichuk, V. F. Zhuravlev, and A. A. Pokutnyi, "Normally solvable operator equations in a Banach space," *Ukr. Mat. Zh.,* **65**, No. 2, 163–174 (2013); ***English translation:*** *Ukr. Math. J.,* **65**, No. 2, 172–192 (2013).
13. V. P. Zhuravl'ov, "Linear boundary-value problems for integral Fredholm equations with degenerate kernels in Banach spaces," *Bukov. Mat. Zh.,* **2**, No. 4, 57–66 (2014).
14. M. M. Popov, "Complementable spaces and some problems of the contemporary geometry of Banach spaces," *Mat. S'ohodni'07,* Issue 13, 78–116 (2007).
15. V. F. Zhuravlev, "Solvability criterion and representation of solutions of n-normal and (d)-normal linear operator equations in a Banach space," *Ukr. Mat., Zh.,* **62**, No. 2, 167–182 (2010); ***English translation:*** *Ukr. Math. J.,* **62**, No. 2, 186–202 (2010).