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DICHOTOMY ON SEMIAXES AND THE SOLUTIONS OF LINEAR SYSTEMS WITH DELAY BOUNDED ON THE ENTIRE AXIS

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By using the theory of generalized inverse operators, we obtain a criterion for the existence and the general form of solutions of linear inhomogeneous functional-differential systems with delay bounded on the entire real axis in the case where the corresponding homogeneous system with delay is exponentially dichotomous on the semiaxes.

Preliminary Information

The conditions under which the problem of solutions of linear inhomogeneous ordinary differential systems of equations bounded on the entire real axis $\mathbb{R}$ is a Fredholm problem were studied in [1, 2]. According to these conditions, the corresponding homogeneous system must be exponentially dichotomous on the semiaxes $\mathbb{R}_-$ and $\mathbb{R}_+$. In [3], the conditions under which the analyzed problems for functional-differential equations with delayed argument are Fredholm problems were established by using the classical Hale results [4]. Similar conditions for the functional-differential equations of "mixed type" were established in [5]. In the present paper, we propose a criterion of existence and the general form of solutions of linear inhomogeneous functional-differential systems with delay bounded on the entire real axis. By using the theory of generalized inverse operators [6, 7], we significantly simplify the proofs of the known results and establish new facts.

Let $BC(\mathbb{R}, \mathbb{R}^n)$ be a Banach space of real vector functions continuous and bounded on $\mathbb{R} = (-\infty, +\infty)$, let $BC^1(\mathbb{R}, \mathbb{R}^n)$ be a Banach space of real continuous vector functions bounded on $\mathbb{R}$ together with their derivatives, let $C := C([-r, 0], \mathbb{R}^n)$ be a space of continuous vector functions with the norm

$$\|x\|_C = \sup_{t \in [-r,0]} |x(t)|, \quad r > 0,$$

and let $\mathcal{L}(C[-r, 0], \mathbb{R}^n)$ be the space of linear bounded operators.

In terms of the notation introduced in [4], we consider a linear functional-differential equation with delay:

$$\dot{x}(t) = L(t)x_t + f(t), \quad t \geq \sigma, \tag{1}$$

with the initial condition $x_\sigma(\theta) = \phi(\theta)$, $\sigma - r \leq \theta \leq \sigma$, where $x(t) \in BC^1(\mathbb{R}, \mathbb{R}^n)$,

$$x_t := x_t(\theta) = x(t + \theta) \in C([-r, 0], \mathbb{R}^n)$$

with respect to the variable $\theta \in [-r, 0]$, $\phi(\theta) \in C([-r, 0], \mathbb{R}^n)$, the operator $L(t) \in \mathcal{L}(C[-r, 0], \mathbb{R}^n)$ is continuous in $t \in \mathbb{R}$, $L(t) \in BC(\mathbb{R}, \mathbb{R}^{n\times n})$, and $f(t) \in BC(\mathbb{R}, \mathbb{R}^n)$. It is known [4, p. 177] that the general solution x_t of

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system (1) can be represented in the form

$$x_t(\sigma, \phi, f) = x_t(\sigma, \phi, 0) + \int_{\sigma}^{t} U_t(\cdot, s) f(s) ds, \quad t \geq \sigma, \tag{2}$$

where the matrix $U(t, s)$ is defined as a solution of the homogeneous equation

$$(Fx)(t) := \dot{x}(t) - L(t)x_t = 0 \tag{3}$$

with the trivial initial conditions

$$\frac{\partial U(t,s)}{\partial t} = L(t)\, U_t(\cdot, s), \quad U(t,s) = \begin{cases} 0 & \text{for} \quad s - r \leq t < s, \\ I & \text{for} \quad t = s, \end{cases}$$

$$U_t(\cdot, s)(\theta) = U(t + \theta, s), \quad -r \leq \theta \leq 0.$$

It is called the fundamental matrix of Eq. (3) [4, p. 175]. If a solution of the homogeneous system (3) is linear in ϕ,

$$x_t(\sigma, \phi, 0) = T(t, \sigma)\phi,$$

then $U_t(\cdot, s)$ can be rewritten in the form

$$U_t(\cdot, s) = T(t, s)X_0,$$

where the translation operator

$$T(t, s) : C([-r, 0], \mathbb{R}^n) \to C([-r, 0], \mathbb{R}^n)$$

is a semigroup for $t \geq s$,

$$X_0 := X_0(\theta) = \begin{cases} 0 & \text{for} \quad -r \leq \theta < 0, \\ I & \text{for} \quad \theta = 0, \end{cases}$$

is a jump function satisfying Conditions 1–8 from [3, pp. 235, 236]. Under these conditions, the operator $T(t, s)$ is linear and strictly continuous [4, p. 177] with respect to t and s and the integral representation (2) takes the form

$$x_t = T(t, \sigma)\phi + \int_{\sigma}^{t} T(t, s)X_0 f(s) ds, \quad t \geq \sigma. \tag{4}$$

Main Result

Assume that the homogenous delay system (3) is exponentially dichotomous on $\mathbb{R}_- = (-\infty, 0]$ and $\mathbb{R}_+ = [0, +\infty)$ with the projectors $\mathcal{P}_\pm(t) \to \mathcal{P}_\pm$ as $t \to \pm\infty$. Consider the problem of existence and construction of solutions $x_t \in C([-r,0],\mathbb{R}^n)$ of the inhomogeneous system (1) bounded on $\mathbb{R}$ in the case where the homogeneous system (3) has nontrivial solutions bounded on $\mathbb{R}$.

It is known [3] that the solution x_t of problem (1) bounded on the semiaxes has the form

$$x_t = T(t,0)\mathcal{P}_+(0)\phi + \int_0^t T(t,s)\mathcal{P}_+(s)X_0 f(s)ds$$

$$- \int_t^\infty T(t,s)[I-\mathcal{P}_+(s)]X_0 f(s)ds, \quad t \in \mathbb{R}_+, \tag{5}$$

$$x_t = T(t,0)[I-\mathcal{P}_-(0)]\phi + \int_{-\infty}^t T(t,s)[I-\mathcal{P}_-(s)]X_0 f(s)ds$$

$$+ \int_0^t T(t,s)\mathcal{P}_-(s)X_0 f(s)ds, \quad t \in \mathbb{R}_-, \tag{6}$$

where $\phi \in C([-r,0],\mathbb{R}^n)$ is an arbitrary element that should be determined from the condition guaranteeing that solutions (5) and (6) are bounded on the entire axis $\mathbb{R}$ if and only if the element $\phi \in C([-r,0],\mathbb{R}^n)$ satisfies the condition

$$x_t(0-,\phi) = x_t(0+,\phi). \tag{7}$$

Thus, substituting (5) and (6) in (7) and taking into account the fact that $T(0,0) = I$, we conclude that the element $\phi \in C([-r,0],\mathbb{R}^n)$ must satisfy the operator equation

$$[\mathcal{P}_+(0) - (I-\mathcal{P}_-(0))]\phi = \int_{-\infty}^0 T(0,s)\mathcal{P}_-(s)X_0 f(s)ds + \int_0^\infty T(0,s)[I-\mathcal{P}_+(s)]X_0 f(s)ds. \tag{8}$$

To solve the operator equation (8), we use the well-developed theory of generalized inverse operators [6, 7]. By

$$D := [\mathcal{P}_+(0) - (I-\mathcal{P}_-(0))] : C([-r,0],\mathbb{R}^n) \to C([-r,0],\mathbb{R}^n)$$

we denote an $(n\times n)$ matrix with constant components. By D^- we denote a matrix generalized inverse to the matrix D. Further, by $\mathcal{P}_{N(D)}$ we denote a finite-dimensional projector of the space C onto the null space $N(D)$ of the operator D, i.e., $\mathcal{P}_{N(D)} : C \to N(D)$ and $\mathcal{P}^2_{N(D)} = \mathcal{P}_{N(D)}$. Moreover, by $\mathcal{P}_{Y_D}$ we denote a finite-dimensional projector of the space C onto the subspace $Y_D = C \ominus R(D)$, i.e., $\mathcal{P}_{Y_D} : C \to Y_D$ and $\mathcal{P}^2_{Y_D} = \mathcal{P}_{Y_D}$. The generalized inverse matrix D^- is connected with projectors $\mathcal{P}_{N(D)}$ and $\mathcal{P}_{Y_D}$ by the formulas [6, 7]

$$DD^- = I - \mathcal{P}_{N(D)}, \quad D^-D = I - \mathcal{P}_{Y_D},$$

where the projectors $\mathcal{P}_{N(D)}$ and $\mathcal{P}_{Y_D}$ are $(n \times n)$ constant matrices. Equation (8) is solvable if and only if

$$\mathcal{P}_{Y_D}\left[\int\limits_{-\infty}^{0} T(0,s)\mathcal{P}_-(s)X_0 f(s)ds + \int\limits_{0}^{\infty} T(0,s)(I-\mathcal{P}_+(s))X_0 f(s)ds\right] = 0. \tag{9}$$

Since

$$\mathcal{P}_{Y_D} D = \mathcal{P}_{Y_D}[\mathcal{P}_+(0) - (I - \mathcal{P}_-(0))] = 0,$$

we have

$$\mathcal{P}_{Y_D}[I - \mathcal{P}_+(0)] = \mathcal{P}_{Y_D}\mathcal{P}_-(0).$$

In view of the relation [3, p. 236]

$$T(t,s)\mathcal{P}(s)X_0 = \mathcal{P}(t)T(t,s)X_0,$$

which takes the form

$$T(0,s)\mathcal{P}(s)X_0 = \mathcal{P}(0)T(0,s)X_0$$

for $t = 0$, condition (9) is equivalent to the conditions

$$\mathcal{P}_{Y_D}\int\limits_{-\infty}^{\infty} \mathcal{P}_-(0)T(0,s)X_0 f(s)ds = 0 \quad \text{or} \quad \mathcal{P}_{Y_D}\int\limits_{-\infty}^{\infty} [I-\mathcal{P}_+(0)]T(0,s)X_0 f(s)ds = 0. \tag{10}$$

Let

$$\operatorname{rang}[\mathcal{P}_{Y_D}\mathcal{P}_-(0)] = \operatorname{rang}[\mathcal{P}_{Y_D}(I - \mathcal{P}_+(0))] = \nu.$$

By

$$_\nu[\mathcal{P}_{Y_D}\mathcal{P}_-(0)] =_\nu [\mathcal{P}_{Y_D}(I - \mathcal{P}_+(0))]$$

we denote a $(\nu \times n)$ matrix whose rows are ν linearly independent rows of the matrix

$$[\mathcal{P}_{Y_D}\mathcal{P}_-(0)] = [\mathcal{P}_{Y_D}(I - \mathcal{P}_+(0))].$$

By H_ν we denote a $(\nu \times n)$ matrix

$$H_\nu(s,0) =_\nu [\mathcal{P}_{Y_D}\mathcal{P}_-(0)]T(0,s) =_\nu [\mathcal{P}_{Y_D}(I - \mathcal{P}_+(0))]T(0,s).$$

Thus, each condition in (10) consists of ν linearly independent conditions

$$\int\limits_{-\infty}^{\infty} H_\nu(s,0)X_0 f(s)ds = 0. \tag{11}$$

Remark 1. The conditions of solvability (11) of Eq. (8) are equivalent to the condition [3, p. 241]

$$\int_{-\infty}^{\infty} y(s)f(s)ds = 0 \tag{12}$$

for all solutions $y(s)$ bounded on the entire real axis of the system formally conjugate to the original system (3) [4, p. 179]. It follows from (11) and (12) that $H_\nu(s,0)$ is a resolving operator of the problem of bounded solutions of a formally conjugate system consisting of ν linearly independent bounded solutions of the conjugate system.

The operator equation (8) is solvable with respect to ϕ if and only if its right-hand side satisfies condition (11) under which the operator equation (8) possesses the solution

$$\phi = \mathcal{P}_{N(D)}\hat{\phi} + D^{-}\left[\int_{-\infty}^{0} T(0,s)\mathcal{P}_{-}(s)X_0 f(s)ds + \int_{0}^{\infty} T(0,s)[I-\mathcal{P}_{+}(s)]X_0 f(s)ds\right], \tag{13}$$

where $\hat{\phi}$ is an arbitrary element from the space $C([-r,0],\mathbb{R}^n)$.

Substituting (13) in (5) and (6), we obtain the following general solution x_t of system (1) bounded on the entire real axis:

$$x_t = T(t,0)\mathcal{P}_{+}(0)\mathcal{P}_{N(D)}\hat{\phi} + \int_{0}^{t} T(t,s)\mathcal{P}_{+}(s)X_0 f(s)ds - \int_{t}^{\infty} T(t,s)[I-\mathcal{P}_{+}(s)]X_0 f(s)ds$$

$$+\, T(t,0)\mathcal{P}_{+}(0)D^{-}\left[\int_{-\infty}^{0} T(0,s)\mathcal{P}_{-}(s)X_0 f(s)ds + \int_{0}^{\infty} T(0,s)[I-\mathcal{P}_{+}(s)]X_0 f(s)ds\right], \quad t\in\mathbb{R}_{+},$$

$$x_t = T(t,0)[I-\mathcal{P}_{-}(0)]\mathcal{P}_{N(D)}\hat{\phi} + \int_{0}^{t} T(t,s)[I-\mathcal{P}_{-}(s)]X_0 f(s)ds$$

$$+ \int_{-\infty}^{t} T(t,s)\mathcal{P}_{-}(s)X_0 f(s)ds + T(t,0)[I-\mathcal{P}_{-}(0)]D^{-}$$

$$\times\left[\int_{-\infty}^{0} T(0,s)\mathcal{P}_{-}(s)X_0 f(s)ds + \int_{0}^{\infty} T(0,s)[I-\mathcal{P}_{+}(s)]X_0 f(s)ds\right], \quad t\in\mathbb{R}_{-}.$$

Since

$$D\mathcal{P}_{N(D)} = [\mathcal{P}_{+}(0) - (I-\mathcal{P}_{-}(0))]\mathcal{P}_{N(D)} = 0,$$

we have

$$\mathcal{P}_{+}(0)\mathcal{P}_{N(D)} = [I-\mathcal{P}_{-}(0)]\mathcal{P}_{N(D)}.$$

Let

$$\operatorname{rang}[\mathcal{P}_+(0)\mathcal{P}_{N(D)}] = \operatorname{rang}[(I-\mathcal{P}_-(0))\mathcal{P}_{N(D)}] = \mu.$$

By $[\mathcal{P}_+(0)\mathcal{P}_{N(D)}]_\mu$ we denote an $(n \times \mu)$ matrix whose columns represent a complete system of linearly independent columns of the matrix $[\mathcal{P}_+(0)\mathcal{P}_{N(D)}]$ and by $[(I-\mathcal{P}_-(0))\mathcal{P}_{N(D)}]_\mu$ we denote an $(n \times \mu)$ matrix whose columns represent a complete system of linearly independent columns of the matrix $[(I-\mathcal{P}_-(0))\mathcal{P}_{N(D)}]$. Then

$$T_\mu(t,0) = T(t,0)[\mathcal{P}_+(0)\mathcal{P}_{N(D)}]_\mu = T(t,0)[(I-\mathcal{P}_-(0))\mathcal{P}_{N(D)}]_\mu \tag{14}$$

is a resolving operator of the problem of solutions of system (3) bounded on the entire axis $\mathbb{R}$. Since the operator $T(t,s)$ forms a semigroup, we get

$$T(t,s) = T(t,0)T(0,s).$$

According to the results presented above, the general solution x_t of the inhomogeneous system (1) bounded on the entire axis $\mathbb{R}$ can be rewritten in the form

$$x_t = T_\mu(t,0)\phi_\mu + T(t,0)\Bigg\{\int_0^t T(0,s)\mathcal{P}_+(s)X_0 f(s)ds - \int_t^\infty T(0,s)[I-\mathcal{P}_+(s)]X_0 f(s)ds$$

$$+ \mathcal{P}_+(0)D^-\Bigg[\int_{-\infty}^0 T(0,s)\mathcal{P}_-(s)X_0 f(s)ds$$

$$+ \int_0^\infty T(0,s)[I-\mathcal{P}_+(s)]X_0 f(s)ds\Bigg]\Bigg\}, \quad t \in \mathbb{R}_+,$$

$$x_t = T_\mu(t,0)\phi_\mu + T(t,0)\Bigg\{\int_0^t T(0,s)[I-\mathcal{P}_-(s)]X_0 f(s)ds + \int_{-\infty}^t T(0,s)\mathcal{P}_-(s)X_0 f(s)ds$$

$$+ [I-\mathcal{P}_-(0)]D^-\Bigg[\int_{-\infty}^0 T(0,s)\mathcal{P}_-(s)X_0 f(s)ds$$

$$+ \int_0^\infty T(0,s)[I-\mathcal{P}_+(s)]X_0 f(s)ds\Bigg]\Bigg\}, \quad t \in \mathbb{R}_-,$$

where ϕ_μ is an arbitrary μ-dimensional column from the space $C([-r,0],\mathbb{R}^\mu)$.

Hence, the following statement is proved:

Theorem 1. *Let an operator F be exponentially dichotomous on the semiaxes $\mathbb{R}_-$ and $\mathbb{R}_+$ with the projectors $\mathcal{P}_\pm(t) \to \mathcal{P}_\pm$ as $t \to \pm\infty$.*

Then the homogeneous system (3) has a μ*-parametric,*

$$\mu = \operatorname{rang}[\mathcal{P}_+(0)\mathcal{P}_{N(D)}] = \operatorname{rang}[(I - \mathcal{P}_-(0))\mathcal{P}_{N(D)}],$$

family of solutions bounded on $\mathbb{R}$

$$x_t = T_\mu(t, 0)\phi_\mu,$$

where $\phi_\mu \in C([-r, 0], \mathbb{R}^\mu)$ *is an arbitrary* μ*-dimensional vector function and* $T_\mu(t, 0)$ *is the resolving operator (14) of the problem of solutions of the homogeneous system (3) bounded on* $\mathbb{R}$.

Under condition (11) and only under this condition, the inhomogeneous problem (1) has a μ*-parametric family of linearly independent solutions bounded on* $\mathbb{R}$

$$x_t = T_\mu(t, 0)\phi_\mu + (Gf)(t), \tag{15}$$

where

$$(Gf)(t) = T(t,0)\begin{cases} \displaystyle\int_0^t T(0,s)\mathcal{P}_+(s)X_0 f(s)ds - \int_t^\infty T(0,s)[I - \mathcal{P}_+(s)]X_0 f(s)ds \\ \quad \displaystyle + \mathcal{P}_+(0)D^-\left[\int_{-\infty}^0 T(0,s)\mathcal{P}_-(s)X_0 f(s)ds\right. \\ \quad \displaystyle \left. + \int_0^\infty T(0,s)[I - \mathcal{P}_+(s)]X_0 f(s)ds\right], \quad t \in \mathbb{R}_+, \\ \displaystyle\int_0^t T(0,s)[I - \mathcal{P}_-(s)]X_0 f(s)ds + \int_{-\infty}^t T(0,s)\mathcal{P}_-(s)X_0 f(s)ds \\ \quad \displaystyle + [I - \mathcal{P}_-(0)]D^-\left[\int_{-\infty}^0 T(0,s)\mathcal{P}_-(s)X_0 f(s)ds\right. \\ \quad \displaystyle \left. + \int_0^\infty T(0,s)[I - \mathcal{P}_+(s)]X_0 f(s)ds\right], \quad t \in \mathbb{R}_-, \end{cases} \tag{16}$$

is the generalized Green operator of the problem of solutions of the inhomogeneous system (1) bounded on $\mathbb{R}$; *this operator satisfies the conditions*

$$(FG[f])(t) = f(t), \quad t \in \mathbb{R},$$

$$(G[f])(0+0) - (G[f])(0-0) = \int_{-\infty}^{\infty} H(s,0)X_0 f(s)ds,$$

where $H(s, 0) = [\mathcal{P}_{Y_D}\mathcal{P}_-(0)]T(0, s) = [\mathcal{P}_{Y_D}(I - \mathcal{P}_+(0))]T(0, s)$.

As an application of Theorem 1, we consider three cases in which the homogeneous system (3) is exponentially dichotomous on $\mathbb{R}_+$ and $\mathbb{R}_-$ with projectors $\mathcal{P}_\pm(t) \to \mathcal{P}_\pm$ as $t \to \pm\infty$ satisfying additional conditions.

Corollary 1. *Let an operator F be exponentially dichotomous on the semiaxes $\mathbb{R}_-$ and $\mathbb{R}_+$ with projectors $\mathcal{P}_\pm(t) \to \mathcal{P}_\pm$ as $t \to \pm\infty$ satisfying the condition*

$$\mathcal{P}_+(0)\mathcal{P}_-(0) = \mathcal{P}_-(0)\mathcal{P}_+(0) = \mathcal{P}_-(0). \tag{17}$$

Then the homogeneous system (3) has a μ-parameter,

$$\mu = \operatorname{rang} \mathcal{P}_{N(D)} = \operatorname{rang} [\mathcal{P}_+(0) - \mathcal{P}_-(0)],$$

family of linearly independent solutions bounded on $\mathbb{R}$:

$$x_t = T_\mu(t, 0)\phi_\mu,$$

where $\phi_\mu \in C([-r, 0], \mathbb{R}^\mu)$ is an arbitrary μ-dimensional vector function.

For any $f(t) \in BC(\mathbb{R}, \mathbb{R}^n)$, the inhomogeneous problem (1) has a μ-parametric family of linearly independent solutions bounded on $\mathbb{R}$

$$x_t = T_\mu(t, 0)\phi_\mu + (Gf)(t),$$

where $(Gf)(t)$ is the generalized Green operator of the problem of solutions of the inhomogeneous system (3) of the form (16) with

$$\mathcal{P}_+(0)D^- = \mathcal{P}_-(0) \quad \text{and} \quad [I - \mathcal{P}_-(0)]D^- = -[I - \mathcal{P}_+(0)]$$

bounded on $\mathbb{R}$; this operator satisfies the conditions

$$(FG[f])(t) = f(t), \quad t \in \mathbb{R},$$

$$(G[f])(0 + 0) - (G[f])(0 - 0) = 0.$$

Proof. Assume that the projectors $\mathcal{P}_+(0)$ and $\mathcal{P}_-(0)$ satisfy condition (17). This case corresponds to the condition of exponential trichotomy of system (3) well known in the theory of ordinary differential equations without delay [10]. We now show that, in this case,

(i) $D^- = D$,

(ii) $\mathcal{P}_{N(D)} = \mathcal{P}_{Y_D} = \mathcal{P}_+(0) - \mathcal{P}_-(0)$.

It is known [7] that the operator D^- is generalized-inverse to the operator D if it satisfies the condition

$$DD^-D = D$$

and, as a corollary, two more additional conditions

$$DD^- = I - \mathcal{P}_{Y_D}, \tag{18}$$

$$D^-D = I - \mathcal{P}_{N(D)}. \tag{19}$$

We first determine the square of the operator D :

$$D^2 = [\mathcal{P}_+(0) - (I - \mathcal{P}_-(0))]^2 = [\mathcal{P}_+(0) - I + \mathcal{P}_-(0)]^2$$
$$= I - \mathcal{P}_+(0) - \mathcal{P}_-(0) + 2\mathcal{P}_+(0)\mathcal{P}_-(0) = I - \mathcal{P}_+(0) + \mathcal{P}_-(0). \tag{20}$$

Further, we determine D^3 :

$$D^3 = [I - \mathcal{P}_+(0) + \mathcal{P}_-(0)]D = [I - \mathcal{P}_+(0) + \mathcal{P}_-(0)][\mathcal{P}_+(0) - (I - \mathcal{P}_-(0))]$$
$$= \mathcal{P}_+(0) - [I - \mathcal{P}_-(0)] = D.$$

Hence, $DDD = D$, i.e., $D^- = D$.

Since $D^2 = DD^-$ and, according to condition (17), $\mathcal{P}_+(0)\mathcal{P}_-(0) = \mathcal{P}_-(0)$, we obtain

$$\mathcal{P}_{N(D)} = I - D^- D = I - D^2 = \mathcal{P}_+(0) - \mathcal{P}_-(0)$$

from equalities (19) and (20) and

$$\mathcal{P}_{Y_D} = I - DD^- = I - D^2 = \mathcal{P}_+(0) - \mathcal{P}_-(0)$$

from equalities (18) and (20).

Hence,

$$\mathcal{P}_{N(D)} = \mathcal{P}_{Y_D} = \mathcal{P}_+(0) - \mathcal{P}_-(0).$$

Since $\mathcal{P}_{Y_D} = \mathcal{P}_-(0) - \mathcal{P}_+(0)$ and, in view of relations (17), $\mathcal{P}_+(0)\mathcal{P}_-(0) = \mathcal{P}_+(0)$, we get

$$\mathcal{P}_{Y_D}\mathcal{P}_-(0) = [\mathcal{P}_-(0) - \mathcal{P}_+(0)]\mathcal{P}_-(0) = \mathcal{P}_-^2(0) - \mathcal{P}_+(0)\mathcal{P}_-(0) = \mathcal{P}_-(0) - \mathcal{P}_+(0) = \mathcal{P}_{Y_D}.$$

Therefore, the condition necessary and sufficient for the solvability (11) of problem (1) of solutions bounded on $\mathbb{R}$ has the form

$$\int_{-\infty}^{\infty} H_\nu(s,0) X_0 f(s) ds = 0,$$

where $H_\nu(s,0) = {}_\nu[\mathcal{P}_{Y_D}]T(0,s)$.

Since

$$\mathcal{P}_{N(D)} = \mathcal{P}_-(0) - \mathcal{P}_+(0) \quad \text{and} \quad \mathcal{P}_+(0)\mathcal{P}_-(0) = \mathcal{P}_+(0),$$

in view of relation (17), we get

$$\mathcal{P}_+(0)\mathcal{P}_{N(D)} = \mathcal{P}_+(0)[\mathcal{P}_-(0) - \mathcal{P}_+(0)] = \mathcal{P}_+(0)\mathcal{P}_-(0)) - \mathcal{P}_+^2(0) = \mathcal{P}_+(0) - \mathcal{P}_+(0) = 0.$$

Hence,

$$T_\mu(t,0) = T(t,0)[\mathcal{P}_+(0)\mathcal{P}_{N(D)}]_\mu = 0$$

and the homogeneous equation (3) possesses only a trivial solution bounded on $\mathbb{R}$.

Since $D^- = D$, we get

$$\mathcal{P}_+(0)D^- = \mathcal{P}_+(0)[\mathcal{P}_+(0) - (I - \mathcal{P}_-(0))] = \mathcal{P}_+^2(0) - \mathcal{P}_+(0)[I - \mathcal{P}_-(0)] = \mathcal{P}_+(0)$$

and

$$[I - \mathcal{P}_-(0)]D^- = [I - \mathcal{P}_-(0)][\mathcal{P}_+(0) - [I - \mathcal{P}_-(0)]] = -[I - \mathcal{P}_+(0)].$$

Corollary 1 is proved.

Corollary 2. *Assume that an operator F is exponentially dichotomous on the semiaxes $\mathbb{R}_-$ and $\mathbb{R}_+$ with projectors $\mathcal{P}_\pm(t) \to \mathcal{P}_\pm$ as $t \to \pm\infty$ and satisfies the condition*

$$\mathcal{P}_+(0)\mathcal{P}_-(0) = \mathcal{P}_-(0)\mathcal{P}_+(0) = \mathcal{P}_+(0). \tag{21}$$

Then the homogeneous system (3) has only a trivial solution bounded on $\mathbb{R}$.

The inhomogeneous problem (1) is solvable for those and only those $f(t) \in BC(\mathbb{R}, \mathbb{R}^n)$ that satisfy the condition

$$\int_{-\infty}^{\infty} H_\nu(s,0)X_0 f(s)ds = 0,$$

and, moreover, possesses a unique solution bounded on $\mathbb{R}$

$$x_t = (Gf)(t),$$

where $(Gf)(t)$ is the generalized Green operator of the problem of solutions of the inhomogeneous system (3) of the form (16) with

$$\mathcal{P}_+(0)D^- = \mathcal{P}_+(0) \quad \text{and} \quad [I - \mathcal{P}_-(0)]D^- = -[I - \mathcal{P}_-(0)]$$

bounded on $\mathbb{R}$ that satisfies the conditions

$$(FG(f))(t) = f(t), \quad t \in \mathbb{R},$$

$$(G(f))(0+0) - (G(f))(0-0) = \int_{-\infty}^{\infty} H(s,0)X_0 f(s)ds.$$

Proof. Assume that the projectors $\mathcal{P}_+(0)$ and $\mathcal{P}_-(0)$ satisfy condition (21). Then:

(i) $D^- = D$,

(ii) $\mathcal{P}_{N(D)} = \mathcal{P}_{Y_D} = \mathcal{P}_-(0) - \mathcal{P}_+(0)$.

Relations (i) and (ii) are proved in exactly the same way as in Corollary 1.

Since $\mathcal{P}_{Y_D} = \mathcal{P}_-(0) - \mathcal{P}_+(0)$ and, in view of relation (21), $\mathcal{P}_+(0)\mathcal{P}_-(0) = \mathcal{P}_+(0)$, we get

$$\mathcal{P}_{Y_D}\mathcal{P}_-(0) = [\mathcal{P}_-(0) - \mathcal{P}_+(0)]\mathcal{P}_-(0) = \mathcal{P}_-^2(0) - \mathcal{P}_+(0)\mathcal{P}_-(0) = \mathcal{P}_-(0) - \mathcal{P}_+(0) = \mathcal{P}_{Y_D}.$$

Therefore, condition (11) necessary and sufficient for the solvability of the solutions of problem (1) bounded on $\mathbb{R}$ has the form

$$\int_{-\infty}^{\infty} H_\nu(s,0)X_0 f(s)ds = 0,$$

where

$$H_\nu(s,0) = {}_\nu[\mathcal{P}_{Y_D}]T(0,s).$$

Since $\mathcal{P}_{N(D)} = \mathcal{P}_-(0) - \mathcal{P}_+(0)$ and, in view of relation (21), $\mathcal{P}_+(0)\mathcal{P}_-(0) = \mathcal{P}_+(0)$, we find

$$\mathcal{P}_+(0)\mathcal{P}_{N(D)} = \mathcal{P}_+(0)[\mathcal{P}_-(0) - \mathcal{P}_+(0)] = \mathcal{P}_+(0)\mathcal{P}_-(0)) - \mathcal{P}_+^2(0) = \mathcal{P}_+(0) - \mathcal{P}_+(0) = 0.$$

Hence,

$$T_\mu(t,0) = T(t,0)[\mathcal{P}_+(0)\mathcal{P}_{N(D)]_\mu} = 0$$

and the homogeneous equation (3) possesses only a trivial solution bounded on $\mathbb{R}$.

Since $D^- = D$, we obtain

$$\mathcal{P}_+(0)D^- = \mathcal{P}_+(0)[\mathcal{P}_+(0) - (I - \mathcal{P}_-(0))] = \mathcal{P}_+^2(0) - \mathcal{P}_+(0)[I - \mathcal{P}_-(0)] = \mathcal{P}_+(0)$$

and

$$[I - \mathcal{P}_-(0)]D^- = [I - \mathcal{P}_-(0)][\mathcal{P}_+(0) - [I - \mathcal{P}_-(0)]] = -[I - \mathcal{P}_-(0)].$$

Corollary 2 is proved.

Corollary 3. *Assume that an operator F is exponentially dichotomous on the semiaxes $\mathbb{R}_-$ and $\mathbb{R}_+$ with projectors $\mathcal{P}_\pm(t) \to \mathcal{P}_\pm$ as $t \to \pm\infty$ and satisfies the condition*

$$\mathcal{P}_+(0)\mathcal{P}_-(0) = \mathcal{P}_-(0)\mathcal{P}_+(0) = \mathcal{P}_+(0) = \mathcal{P}_-(0). \tag{22}$$

Then the homogeneous system (3) is exponentially dichotomous on $\mathbb{R}$ and possesses only a trivial solution bounded on $\mathbb{R}$.

For any $f(t) \in BC(\mathbb{R}, \mathbb{R}^n)$, the inhomogeneous problem (1) possesses a unique solution bounded on $\mathbb{R}$

$$x_t = (Gf)(t),$$

where $(Gf)(t)$ is the Green operator of the problem of solutions of the inhomogeneous system (3) of the form (16) with

$$\mathcal{P}_+(0)D^- = \mathcal{P}_+(0) \quad \textit{and} \quad [I - \mathcal{P}_-(0)]D^- = -[I - \mathcal{P}_+(0)]$$

bounded on $\mathbb{R}$.

Proof. Assume that the homogeneous system (3) is exponentially dichotomous on $\mathbb{R}_+$ and $\mathbb{R}_-$ with projectors $\mathcal{P}_\pm(t) \to \mathcal{P}_\pm$ as $t \to \pm\infty$ such that condition (22) is satisfied.

Since condition (22) is satisfied, the matrix D has the form

$$D = \mathcal{P}_+(0) - [I - \mathcal{P}_-(0)] = \mathcal{P}_+(0) - I + \mathcal{P}_-(0) = \mathcal{P}_+(0) - I + \mathcal{P}_+(0) = 2\mathcal{P}_+(0) - I = J,$$

where J is an involution and $J^2 = I$. Therefore, $D^2 = I$, which means that $D^{-1} = D$.

In view of relation (22), $\mathcal{P}_-(0) = \mathcal{P}_+(0)$, which yields $\mathcal{P}_{Y_D} = \mathcal{P}_{N(D)} = 0$. Hence,

$$\mathcal{P}_+(0)\mathcal{P}_{N(D)} = 0 \quad \text{and} \quad \mathcal{P}_{Y_D}\mathcal{P}_-(0) = 0.$$

Thus, the condition (11) necessary and sufficient for the solvability of the problem (1) of solutions bounded on $\mathbb{R}$ is satisfied for all $f(t)$, the homogeneous equation (3) possesses only the trivial solution bounded on $\mathbb{R}$, and the inhomogeneous system (1) possesses a unique solution bounded on $\mathbb{R}$ for any $f(t) \in BC(\mathbb{R}, \mathbb{R}^n)$.

Remark 2. The established assertions with the corresponding complements and modifications remain true in the case where the operator $L(t)$ and the function $f(t)$ are piecewise continuous with finitely many discontinuities of the first kind with respect to t and bounded on $\mathbb{R}$.

We illustrate this conclusion by the following examples:

1. Consider a linear differential system with constant delay

$$\begin{aligned} \dot{x}(t) &= Lx(t-1) + f(t), \quad t \geq 0, \\ x_0(\theta) &= \phi(\theta), \quad -1 \leq \theta \leq 0, \end{aligned} \tag{23}$$

where L is a matrix of the form

$$L = \begin{cases} L_+ = \operatorname{diag}\{-e^{-1}, e, -e^{-1}\} & \text{for} \quad t \geq 0, \\ L_- = \operatorname{diag}\{-e^{-1}, -e^{-1}, e\} & \text{for} \quad t < 0, \end{cases}$$

and

$$f(t) = \begin{cases} f_+(t) = \operatorname{col}\{f_+^{(1)}(t), f_+^{(2)}(t), f_+^{(3)}(t)\} & \text{for} \quad t \geq 0, \\ f_-(t) = \operatorname{col}\{f_-^{(1)}(t), f_-^{(2)}(t), f_-^{(3)}(t)\} & \text{for} \quad t < 0, \end{cases}$$

is a function with components continuous and bounded on the corresponding intervals and a discontinuity of the first kind for $t = 0$.

The fundamental matrix $U(t)$ on the intervals has the form

$$U(t) = \begin{cases} U_+(t) = \mathrm{diag}\,\{e^{-t}, e^{t}, e^{-t)}\} & \text{for} \quad t \ge 0, \\ U_-(t) = \mathrm{diag}\,\{e^{-t}, e^{-t}, e^{t}\} & \text{for} \quad t < 0. \end{cases}$$

Then, in the notation introduced in [4], we can rewrite the solution x_t in the form

$$x_t = T(t,0)\phi + \int\limits_0^t T(t,s) X_0 f(s) ds, \tag{24}$$

where the operator $T(t,s)$ admits the representation

$$T(t,s) = \begin{cases} T_+(t,s) = \mathrm{diag}\,\{e^{-(t-1-s)}, e^{t-1-s}, e^{-(t-1-s)}\} & \text{for} \quad t \ge 0, \\ T_-(t,s) = \mathrm{diag}\,\{e^{-(t-1-s)}, e^{-(t-1-s)}, e^{t-1-s}\} & \text{for} \quad t < 0, \end{cases}$$

and

$$X_0 = \mathrm{diag}\,\{X_0^{(1)}, X_0^{(2)}, X_0^{(3)}\}, \quad i = 1,2,3; \quad X_0^{(i)} = X_0^{(i)}(\theta) = \begin{cases} 0 & \text{for} \quad -1 \le \theta < 0, \\ 1 & \text{for} \quad \theta = 0. \end{cases}$$

The homogeneous system corresponding to (23) is exponentially dichotomous on the semiaxes $\mathbb{R}_+$ and $\mathbb{R}_-$ with the projectors

$$\mathcal{P}_+(0) = \mathrm{diag}\,\{1,0,1\} \quad \text{and} \quad \mathcal{P}_-(0) = \mathrm{diag}\,\{1,1,0\},$$

respectively. Thus,

$$D = \mathcal{P}_+(0) - [I - \mathcal{P}_-(0)] = \mathrm{diag}\,\{1,0,0\},$$

$$D^- = \mathrm{diag}\,\{1,0,0\}.$$

The projectors $\mathcal{P}_{N(D)} : \mathbb{R}^3 \to N(D)$ and $\mathcal{P}_{Y_D} : \mathbb{R}^3 \to Y_D$ are identical:

$$\mathcal{P}_{N(D)} = \mathcal{P}_{Y_D} = \mathrm{diag}\{0,1,1\}.$$

The ranks of the matrices

$$[\mathcal{P}_{Y_D}\mathcal{P}_-(0)] = [\mathcal{P}_{Y_D}(I - \mathcal{P}_+(0))] = \mathrm{diag}\,\{0,1,0\}$$

are equal to 1. By

$${}_1[\mathcal{P}_{Y_D}\mathcal{P}_-(0)] = {}_1\,[\mathcal{P}_{Y_D}(I - \mathcal{P}_+(0))] = [\ 0 \quad 1 \quad 0\ ]$$

we denote a (1×3) matrix. Then the condition of existence of a solution of the inhomogeneous system (23) bounded on the entire axis takes the form

$$[\,0 \quad 1 \quad 0\,]\left[\int_{-\infty}^{0} T_{-}(0,s)X_0 f(s)ds + \int_{0}^{\infty} T_{+}(0,s)X_0 f(s)ds\right] = 0. \tag{25}$$

Since

$$X_0 = \begin{cases} 0 & \text{for } -1 \le \theta < 0, \\ I & \text{for } \theta = 0, \end{cases}$$

we conclude that $T(t,s)X_0 = 0$ for $t-1 \le s < t$. Hence, for $t = 0$, we get $T(0,s)X_0 = 0$ if $-1 \le s < 0$. By using this result, we arrive at the following relation from (25):

$$\int_{-\infty}^{-1} e^{-(1+s)} f_{-}^{(2)}(s)ds + \int_{0}^{\infty} e^{1+s} f_{+}^{(2)}(s)ds = 0. \tag{26}$$

Under condition (26), the inhomogeneous system (23) has a one-parameter family of bounded solutions of the form (15). Indeed, the ranks of the matrices

$$[\mathcal{P}_{+}(0)\mathcal{P}_{N(D)}] = [(I - \mathcal{P}_{-}(0))\mathcal{P}_{N(D)}] = \operatorname{diag}\{0,0,1\}$$

are equal to 1, i.e., $\mu = 1$. Hence,

$$[\mathcal{P}_{+}(0)\mathcal{P}_{N(D)}]_1 = [(I - \mathcal{P}_{-}(0))\mathcal{P}_{N(D)}]_1 = \operatorname{diag}\{0,0,1\}$$

is a (3×1) matrix and, therefore,

$$T_1(t,0) = \begin{cases} \operatorname{col}\{0,0,e^{-(t-1-s)}\} & \text{for } \ t \ge 0, \\ \operatorname{col}\{0,0,e^{t-1-s}\} & \text{for } \ t < 0. \end{cases} \tag{27}$$

2. Under the same assumptions, we now consider the linear differential system with constant delay (23), where

$$L = \begin{cases} L_{+} = \operatorname{diag}\{-e^{-1}, e, -e^{-1}\} & \text{for } \ t \ge 0, \\ L_{-} = \operatorname{diag}\{-e^{-1}, e, e\} & \text{for } \ t < 0. \end{cases}$$

The fundamental matrix $U(t)$ on the intervals takes the form

$$U(t) = \begin{cases} U_{+}(t) = \operatorname{diag}\{e^{-t}, e^{t}, e^{-t)}\} & \text{for } \ t \ge 0, \\ U_{-}(t) = \operatorname{diag}\{e^{-t}, e^{t}, e^{t}\} & \text{for } \ t < 0. \end{cases}$$

Then the operator $T(t,s)$ admits the representation

$$T(t,s) = \begin{cases} T_+(t,s) = \operatorname{diag}\{e^{-(t-1-s)}, e^{t-1-s}, e^{-(t-1-s)}\} & \text{for} \quad t \geq 0, \\ T_-(t,s) = \operatorname{diag}\{e^{-(t-1-s)}, e^{t-1-s}, e^{t-1-s}\} & \text{for} \quad t < 0. \end{cases}$$

The homogeneous system corresponding to (23) is exponentially dichotomous on the semiaxes $\mathbb{R}_+$ and $\mathbb{R}_-$ with the projectors

$$\mathcal{P}_+(0) = \operatorname{diag}\{1,0,1\} \quad \text{and} \quad \mathcal{P}_-(0) = \operatorname{diag}\{1,0,0\},$$

respectively. Then

$$D = \mathcal{P}_+(0) - [I - \mathcal{P}_-(0)] = \operatorname{diag}\{1,-1,0\},$$

$$D^- = \operatorname{diag}\{1,-1,0\}.$$

The projectors $\mathcal{P}_{N(D)} : \mathbb{R}^3 \to N(D)$ and $\mathcal{P}_{Y_D} : \mathbb{R}^3 \to Y_D$ are identical:

$$\mathcal{P}_{N(D)} = \mathcal{P}_{Y_D} = \operatorname{diag}\{0,0,1\}.$$

The projectors $\mathcal{P}_+(0)$ and $\mathcal{P}_-(0)$ satisfy condition (17).

The matrices

$$[\mathcal{P}_{Y_D}\mathcal{P}_-(0)] = [\mathcal{P}_{Y_D}(I - \mathcal{P}_+(0))] = \operatorname{diag}\{0,0,0\}.$$

Hence, the condition for the existence of a solution of the inhomogeneous system (23) bounded on $\mathbb{R}$ is satisfied for any $f(t) \in BC(\mathbb{R}, \mathbb{R}^n)$. In this case, system (23) has a one-parameter family of bounded solutions (15), where $T_1(t,0)$ has the form (27).

3. Consider a linear differential system with constant delay (23), where

$$L = \begin{cases} L_+ = \operatorname{diag}\{-e^{-1}, e, e\} & \text{for} \quad t \geq 0, \\ L_- = \operatorname{diag}\{-e^{-1}, e, -e^{-1}\} & \text{for} \quad t < 0. \end{cases}$$

The fundamental matrix $U(t)$ on the intervals takes the form

$$U(t) = \begin{cases} U_+(t) = \operatorname{diag}\{e^{-t}, e^{t}, e^{t}\} & \text{for} \quad t \geq 0, \\ U_-(t) = \operatorname{diag}\{e^{-t}, e^{t}, e^{-t}\} & \text{for} \quad t < 0. \end{cases}$$

Then the operator $T(t,s)$ admits the representation

$$T(t,s) = \begin{cases} T_+(t,s) = \operatorname{diag}\{e^{-(t-1-s)}, e^{t-1-s}, e^{t-1-s}\} & \text{for} \quad t \geq 0, \\ T_-(t,s) = \operatorname{diag}\{e^{-(t-1-s)}, e^{t-1-s}, e^{-(t-1-s)}\} & \text{for} \quad t < 0. \end{cases}$$

The homogeneous system corresponding to (23) is exponentially dichotomous on the semiaxes $\mathbb{R}_+$ and $\mathbb{R}_-$ with the projectors

$$\mathcal{P}_+(0) = \operatorname{diag}\{1, 0, 0\} \quad \text{and} \quad \mathcal{P}_-(0) = \operatorname{diag}\{1, 0, 1\},$$

respectively. Then

$$D = \mathcal{P}_+(0) - [I - \mathcal{P}_-(0)] = \operatorname{diag}\{1, -1, 0\},$$

$$D^- = \operatorname{diag}\{1, -1, 0\}.$$

The projectors $\mathcal{P}_+(0)$ and $\mathcal{P}_-(0)$ satisfy condition (21).

The projectors $\mathcal{P}_{N(D)} : \mathbb{R}^3 \to N(D)$ and $\mathcal{P}_{Y_D} : \mathbb{R}^3 \to Y_D$ are identical:

$$\mathcal{P}_{N(D)} = \mathcal{P}_{Y_D} = \operatorname{diag}\{0, 0, 1\}.$$

The ranks of the matrices

$$[\mathcal{P}_{Y_D}\mathcal{P}_-(0)] = [\mathcal{P}_{Y_D}(I - \mathcal{P}_+(0))] = \operatorname{diag}\{0, 0, 1\}$$

are equal to 1. By

$${}_1[\mathcal{P}_{Y_D}\mathcal{P}_-(0)] = {}_1\,[\mathcal{P}_{Y_D}(I - \mathcal{P}_+(0))] = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

we denote a (1×3) matrix. Then the condition for the existence of solutions of the inhomogeneous system (23) bounded on the entire axis takes the form

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \left[\int\limits_{-\infty}^{0} T_-(0, s) X_0 f(s) ds + \int\limits_{0}^{\infty} T_+(0, s) X_0 f(s) ds \right] = 0$$

or, after necessary transformations,

$$\int\limits_{-\infty}^{-1} e^{-(1+s)} f_-^{(3)}(s)\, ds + \int\limits_{0}^{\infty} e^{1+s} f_+^{(3)}(s)\, ds = 0.$$

Since, in this example,

$$[\mathcal{P}_+(0)\mathcal{P}_{N(D)}] = [(I - \mathcal{P}_-(0))\mathcal{P}_{N(D)}]$$

are null matrices, we conclude that $T_\mu(t, s) \equiv 0$ and system (23) possesses a unique solution bounded on the entire axis.

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