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BOUNDARY-VALUE PROBLEMS FOR INTEGRAL EQUATIONS WITH DEGENERATE KERNEL

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We consider boundary-value problems for integral equations with degenerate kernel. Using a pseudoinverse operator, we establish conditions for the existence of a unique solution of the original integral equation and obtain a representation for this solution. We also establish conditions for the existence of a solution of a boundary-value problem for this equation and obtain a representation of this solution. The results are illustrated by examples.

The investigation of the solvability of linear boundary-value problems

$$(Lx)(t) = f(t), \tag{1}$$

$$\ell x(\cdot) = \alpha, \tag{2}$$

where L is a linear bounded operator and ℓ is a linear bounded functional, and the construction of their solutions depend on the solvability of the original operator equation (1). In the case where the operator L is everywhere solvable [1], solvability conditions and formulas for solutions were obtained for many boundary-value problems [2–4]. However, there are many boundary-value problems of the form (1), (2) in which the operator L is not everywhere solvable. Integral equations with degenerate kernel belong to problems of exactly this type. The integral operator that determines the original equation is a Fredholm operator [5, 6] with nonzero kernel. This means that the operator does not have an inverse, and the original equation is solvable not for any right-hand side [1]. The noninvertibility of the original operator substantially complicates the investigation of these boundary-value problems.

The methods of generalized inversion and pseudoinversion of Fredholm and Noetherian operators [4, 7] enable one to solve this problem. We use these methods to find a solvability criterion and construct solutions for normally solvable operator equations and boundary-value problems in the case where the operator L of the original equation is a Fredholm operator with nonzero kernel.

Statement of the Problem

Consider the linear boundary-value problem

$$(Lx)(t) = f(t), \quad t \in [a, b], \tag{3}$$

$$\ell x(\cdot) = \alpha, \tag{4}$$

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where $L: \mathbf{L}_2^n[a,b] \to \mathbf{L}_2^n[a,b]$ is the integral operator of the second kind with the degenerate kernel

$$(Lx)(t) \equiv x(t) - \Psi(t) \int_a^b \Phi(s)x(s)\,ds \tag{5}$$

and $\ell = \operatorname{col}(\ell_1, \ell_2, \ldots, \ell_k): \mathbf{L}_2^n[a,b] \to \mathbf{R}^k$ is a k-dimensional vector functional acting from the space of functions $\mathbf{L}_2^n[a,b]$ square summable on the segment $[a,b]$ into the k-dimensional vector space $\mathbf{R}^k$. The columns of the $n \times m$ matrix $\Psi(t)$, the rows of the $m \times n$ matrix $\Phi(t)$, and the vectors $x(t)$ and $f(t)$ belong to the Hilbert space $\mathbf{L}_2^n[a,b]$. We pose the following problem: Find conditions for the solvability of the operator equation (3) and boundary-value problem (3), (4) and determine representations for their solutions by using the operator L^+ pseudoinverse to L.

Preliminary Information

It is known [5] that the integral operator (5) is a Fredholm operator $(\dim N(L) = \dim N(L^*) = s < \infty)$. We now present, in the general form, one of methods for the construction of an operator pseudoinverse to a Fredholm operator acting from a real Hilbert space $\mathbf{H}_1$ into a real Hilbert space $\mathbf{H}_2$.

Let

$$(x_1(t), x_2(t)) = \int_a^b x_1^*(t)x_2(t)\,dt$$

be the scalar product of an n-dimensional vector column $x_1(t)$ and an n-dimensional vector column $x_2(t)$ in the space $\mathbf{H}_1$, where $*$ denotes transposition. We define the scalar product of an $n \times r$ matrix $X(t)$ and an n-dimensional vector column $x(t)$ by the formula

$$(X(t), x(t)) = \int_a^b X^*(t)x(t)\,dt;$$

as a result, one obtains an r-dimensional column vector of constants. The scalar product of an $n \times m$ matrix $X(t)$ and an $n \times m$ matrix $Y(t)$ is defined by the formula

$$(X(t), Y(t)) = \int_a^b X^*(t)Y(t)\,dt;$$

as a result, one obtains a constant $m \times m$ matrix.

Let $\{f_i\}_{i=1}^s$ be a basis of the null space $N(L)$ of the operator L and let $\{\varphi_j\}_{j=1}^s$ be a basis of the null space $N(L^*)$ of the operator L^* adjoint to L. Using the basis vectors of the null spaces $N(L)$ and $N(L^*)$, we compose the $n \times s$ matrices

$$X(t) = (f_1(t), f_2(t), \ldots, f_s(t)),$$

$$Y(t) = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_s(t))^*.$$

Using relations (3.4) [4, p. 62], we construct orthoprojectors $P_{N(L)}\colon \mathbf{H}_1 \to N(L)$ and $P_{N(L^*)}\colon \mathbf{H}_2 \to N(L^*)$ as follows:

$$(P_{N(L)}x)(t) = X(t)\alpha^{-1}(X^*(t), x(t))_{\mathbf{H}_1}, \tag{6}$$

$$(P_{N(L^*)}y)(t) = Y(t)\beta^{-1}(Y^*(t), y(t))_{\mathbf{H}_2}, \tag{7}$$

where $(\cdot,\cdot)_{\mathbf{H}_1}$ and $(\cdot,\cdot)_{\mathbf{H}_2}$ denote the scalar products in the spaces $\mathbf{H}_1$ and $\mathbf{H}_2$, respectively, and α^{-1} and β^{-1} are the matrices inverse to the symmetric Gram matrices

$$\alpha = (X^*(t), X(t))_{\mathbf{H}_1} \quad \text{and} \quad \beta = (Y^*(t), Y(t))_{\mathbf{H}_2}.$$

If the bases of the null spaces $N(L)$ and $N(L^*)$ are orthonormal, then α and β are identity matrices.

Consider the operators

$$\begin{gathered}(\overline{P}_{N(L^*)}x)(t) = Y(t)\alpha^{-1}(X^*(t), x(t))_{\mathbf{H}_1}, \ \overline{P}_{N(L^*)}\colon \mathbf{H}_1 \to N(L^*), \\ (\overline{P}_{N(L)}y)(t) = X(t)\beta^{-1}(Y^*(t), y(t))_{\mathbf{H}_2}, \ \overline{P}_{N(L)}\colon \mathbf{H}_2 \to N(L).\end{gathered} \tag{8}$$

The operator $\overline{P}_{N(L^*)}$ is an extension (to the space $\mathbf{H}_1$) of an operator that realizes an isomorphism of $N(L)$ onto $N(L^*)$. The operator $\overline{P}_{N(L)}$ is an extension of the inverse of the isomorphic operator to the space $\mathbf{H}_2$.

Lemma 1. *The operator* $\overline{L} = L + \overline{P}_{N(L^*)}$ *has a bounded inverse* $\overline{L}^{-1}$.

The lemma is proved by analogy with the lemma presented in [4].

Using Lemma 1, we prove the following statement:

Theorem 1. *The operator*

$$L^+ = \overline{L}^{-1} - \overline{P}_{N(L)} \tag{9}$$

is a bounded pseudoinverse of the bounded Fredholm operator L.

The proof of the theorem reduces to the verification of the relations that define a unique pseudoinverse operator [7].

Relation (9), which gives a representation of the unique pseudoinverse of a Fredholm operator in a Hilbert space, enables one to find solvability conditions and obtain a representation of a general solution for an integral equation of the second kind with degenerate kernel.

A Criterion for Solvability of Integral Equations with Degenerate Kernel. An Operator Pseudoinverse to an Integral Operator

Assume that the homogeneous equation

$$(Lx)(t) \equiv x(t) - \Psi(t)\int_a^b \Phi(s)x(s)\,ds = 0 \tag{10}$$

has nontrivial solutions, i.e., that Eq. (3) is not everywhere solvable [1]. Let us find conditions for the solvability of Eq. (3) and the general form of its solution under the assumptions made above.

For the construction of an operator pseudoinverse to the integral operator L, we construct bases of the kernels $N(L)$ and $N(L^*)$ of the operators L and L^*, respectively. To this end, we determine general solutions of the homogeneous equation (10) and its conjugate equation

$$(L^* y)(t) \equiv y(t) - \Phi^*(t) \int_a^b \Psi^*(s) y(s)\, ds = 0. \tag{11}$$

Let us find bases of the null spaces of the operators L and L^*. For this purpose, it is necessary to solve the homogeneous equations (10) and (11). We seek a solution of these equations in the form

$$\begin{aligned} x(t) &= \Psi(t)\, c, \\ y(t) &= \Phi^*(t)\, d, \end{aligned} \tag{12}$$

where $c = \operatorname{col}[c_1, c_2, \dots, c_m]$ and $d = \operatorname{col}[d_1, d_2, \dots, d_m]$.

Substituting (12) into (10) and (11), respectively, we obtain the following algebraic system for c and d:

$$\begin{aligned} D\, c &= 0, \\ D^* d &= 0, \end{aligned} \tag{13}$$

where $D = A - E$ $(D^* = A^* - E)$ and

$$A = \int_a^b \Phi(t)\Psi(t)\, dt$$

is a constant $m \times m$ matrix.

Let $P_{N(D)}\colon \mathbf{R}^m \to N(D)$ and $P_{N(D^*)}\colon \mathbf{R}^m \to N(D^*)$ be orthoprojectors [4] to the null spaces $N(D)$ and $N(D^*)$ of the matrices D and D^*, respectively.

The algebraic system (13) has nonzero solutions if and only if $P_{N(D)} \neq 0$, which necessarily yields $P_{N(D^*)} \neq 0$. These conditions are equivalent to the condition that $\det D = 0$, which is assumed in what follows.

Let $\operatorname{rank} D = m - r$ $(\operatorname{rank} D^* = m - r)$. Then each of the $m \times m$ matrix orthoprojectors $P_{N(D)}$ and $P_{N(D^*)}$ has r linearly independent columns. Using these columns, we construct the $n \times r$ matrices $P_{N_r(D)}$ and $P_{N_r(D^*)}$, respectively.

With the use of these matrices, the general solutions of system (13) can be represented in the form [4]

$$\begin{aligned} c &= P_{N_r(D)} c_r, \\ d &= P_{N_r(D^*)} d_r, \end{aligned} \tag{14}$$

where $c_r \in \mathbf{R}^r$ and $d_r \in \mathbf{R}^r$ are arbitrary r-dimensional constant vectors.

Substituting (14) into (12), we obtain the general solutions of the homogeneous integral equations (10) and (11):

$$x(t) = X_r(t)c_r,$$

$$y(t) = Y_r(t)d_r,$$

where $X_r(t) = \Psi(t)P_{N_r(D)}$ and $Y_r(t) = \Phi^*(t)P_{N_r(D^*)}$ are $n \times r$ fundamental matrices whose columns form bases of the null spaces $N(L)$ and $N(L^*)$ of the operators L and L^*.

Using relations (7) and (8), we construct an orthoprojector $P_{N(L^*)}$ and operators $\bar{P}_{N(L)}$ and $\bar{P}_{N(L)}$:

$$(P_{N(L^*)}y)(t) = Y_r(t)\beta^{-1}\int_a^b Y_r^*(s)y(s)\,ds, \quad P_{N(L^*)}\colon \mathbf{L}_2^n[a,b] \to N(L^*),$$

$$(\bar{P}_{N(L^*)}x)(t) = Y_r(t)\alpha^{-1}\int_a^b X_r^*(s)x(s)\,ds, \quad \bar{P}_{N(L^*)}\colon \mathbf{L}_2^n[a,b] \to N(L^*),$$

$$(\bar{P}_{N(L)}y)(t) = X_r(t)\beta^{-1}\int_a^b Y_r^*(s)y(s)\,ds, \quad \bar{P}_{N(L)}\colon \mathbf{L}_2^n[a,b] \to N(L).$$

Here, α^{-1} and β^{-1} are the matrices inverse to the $r \times r$ symmetric Gram matrices

$$\alpha = \int_a^b X_r^*(t)X_r(t)\,dt \quad \text{and} \quad \beta = \int_a^b Y_r^*(t)Y_r(t)\,dt.$$

Then, by virtue of Lemma 1, the operator

$$\left(L + \bar{P}_{N(L^*)}\right)x(t) \equiv x(t) - \Psi(t)\int_a^b \Phi(s)x(s)\,ds + Y_r(t)\beta^{-1}\int_a^b X_r^*(s)x(s)\,ds$$

has a bounded inverse, i.e., the integral equation

$$([L + \bar{P}_{N(L^*)}]x)(t) = f(t) \tag{15}$$

is solvable for any right-hand side.

To find a solution of this equation, we rewrite (15) in the form

$$(\bar{L}x)(t) = [(L + \bar{P}_{N(L^*)})x](t) \equiv x(t) - \Psi_1(t)\int_a^b \Phi_1^*(s)x(s)\,ds = f(t), \tag{16}$$

where $\Psi_1(t) = [\Psi(t), -Y_r(t)\alpha^{-1}]$ is the $n \times (m+r)$ matrix composed of the matrices $\Psi(t)$ and $-Y_r(t)\alpha^{-1}$, and $\Phi_1(t) = [\Phi^*(t), X_r(t)]$ is the $n \times (m+r)$ matrix composed of the matrices $\Phi(t)$ and $X_r^*(t)$.

Following [5], we can represent a solution of Eq. (16) in the form

$$x(t) = ((L + \bar{P}_{N(L^*)} f)(t) \equiv f(t) + \Psi_1(t) M^{-1} \int_a^b \Phi_1^*(s) f(s)\, ds, \tag{17}$$

where M^{-1} is an $(m+r) \times (m+r)$ matrix inverse to the matrix $M = I - B$, where

$$B = \int_a^b \Phi_1^*(t) \Psi_1(t)\, dt.$$

Then, by virtue of Theorem 1, the pseudoinverse of the operator L has the form

$$(L^+ f)(t) = ((L + \bar{P}_{N(L^*)})^{-1} - \bar{P}_{N(L)}) f)(t)$$

$$= f(t) + \Psi_1(t) M^{-1} \int_a^b \Phi_1^*(s) f(s)\, ds - X_r(t)\beta^{-1} \int_a^b Y_r^*(s) f(s)\, ds. \tag{18}$$

Using the properties of the constructed pseudoinverse operator L^+ and the fact that the orthoprojectors $P_{N(L)}$ and $P_{N(L^*)}$ induce a decomposition of the Hilbert space $\mathbf{L}_2^n[a,b]$ into direct sums of mutually orthogonal subspaces $\mathbf{L}_2^n[a,b] = N(L) \oplus R(L^*)$ and $\mathbf{L}_2^n[a,b] = N(L^*) \oplus R(L)$, one can easily prove the following statement:

Theorem 2. *Let* $\operatorname{rank} D = m - r$, $r \neq 0$. *Then the integral Fredholm equation with degenerate kernel (3) is solvable for those and only those* $f(t) \in \mathbf{L}_2^n[a,b]$ *for which*

$$(P_{N(L^*)} f)(t) = Y_r(t)\beta^{-1} \int_a^b Y_r^*(s) f(s)\, ds = 0; \tag{19}$$

in this case, it has an r*-parameter family of solutions of the form*

$$x(t) = X_r(t) c_r + (L^+ f)(t), \tag{20}$$

where the first term is a general solution of the corresponding homogeneous equation, the second term $(L^+ f)(t)$ *is the unique particular solution (18) of Eq. (3) orthogonal to any solution of the homogeneous equation (10), and* $c_r \in \mathbf{R}^r$ *is an arbitrary* r*-dimensional column vector of constants.*

In what follows, we represent solutions of Eq. (3), if they exist, in the form

$$x(t) = X_r(t) c_r + f(t) + \Psi_2(t) \int_a^b \Phi_2^*(s) f(s)\, ds, \tag{21}$$

where $\Psi_2(t) = [\Psi_1(t)M^{-1}, -X_r(t)\beta^{-1}]$ is the $n \times (m+2r)$ matrix composed of the matrices $\Psi_1(t)M^{-1}$ and $X_r(t)\beta^{-1}$, and $\Phi_2(t) = [\Phi_1(t), Y_r(t)]$ is the $n \times (m+2r)$ matrix composed of the matrices $\Phi_1(t)$ and $Y_r(t)$.

By virtue of the linear independence of columns of the matrix $Y_r(t)\beta^{-1}$, the solvability condition (19) is equivalent to the following:

$$\int_a^b Y_r^*(s) f(s)\, ds = 0.$$

Linear Boundary-Value Problems for Integral Equations of the Second Kind with Degenerate Kernel

Let us find solutions of Eq. (3) that satisfy the boundary conditions (4).

Consider problem (3), (4) under the assumption that the corresponding homogeneous boundary-value problem

$$(Lx)(t) = 0, \qquad \ell x(\cdot) = 0 \tag{22}$$

has nontrivial solutions. Under the conditions of Theorem 2, Eq. (3) has solutions that satisfy conditions (4) if and only if the algebraic system

$$Qc_r = \alpha - \ell\left\{ f(\cdot) + \Psi_2(\cdot) \int_a^b \Phi_2^*(s) f(s)\, ds \right\}, \tag{23}$$

which is obtained by the substitution of solution (21) of the integral equation (3) into the integral equations (4), is solvable with respect to c_r. Here, $Q = \ell X_r(\cdot)$ is a constant $k \times r$ matrix.

Let Q^+ be the unique $r \times k$ matrix pseudoinverse to Q [4, 7], let $P_{N(Q)}: \mathbf{R}^r \to N(Q)$ be an $r \times r$ matrix orthoprojector, and let $P_{N(Q^*)}: \mathbf{R}^k \to N(Q^*)$ be a $k \times k$ matrix orthoprojector.

Let $P_{N_\rho(Q)}$ denote a $k \times \rho$ matrix whose columns are ρ linearly independent columns of the matrix $P_{N(Q)}$ $(\rho = k - n_1, n_1 = \text{rank}\, Q)$ and let $P_{N_d(Q^*)}$ be a $d \times r$ matrix whose rows are d linearly independent rows of the matrix $P_{N(Q^*)}$ $(d = r - n_1)$.

The algebraic system (23) is solvable if and only if the following condition satisfied [2, 4]:

$$P_{N_d(Q^*)} \left\{ \alpha - \ell[f(\cdot) + \Psi_2(\cdot) \int_a^b \Phi_2^*(s) f(s)\, ds] \right\} = 0;$$

in this case, it has the ρ-parameter family of solutions

$$c_r = P_{N_\rho(Q)} c_\rho - Q^+ \left\{ \alpha - \ell\left[f(\cdot) + \Psi_2(\cdot) \int_a^b \Phi_2^*(s) f(s)\, ds \right] \right\}, \quad c_\rho \in \mathbf{R}^\rho. \tag{24}$$

Substituting (24) into the general solution (21) of Eq. (3), we obtain the general solution of the boundary-value problem (3), (4):

$$x(t) = X_r(t)P_{N_\rho(Q)}c_\rho + X_r(t)Q^+\alpha - X_r(t)Q^+\ell\left[f(\cdot) + \Psi_2(\cdot)\int_a^b \Phi_2^*(s)f(s)\,ds\right]$$

$$+ f(t) + \Psi_2(t)\int_a^b \Phi_2^*(s)f(s)\,ds$$

$$= X_\rho(t)c_\rho + f(t) + X_r(t)Q^+[\alpha - \ell f(\cdot)] + [\Psi_2(t) - X_r(t)Q^+\ell\Psi_2(\cdot)]\int_a^b \Phi_2^*(s)f(s)ds.$$

Thus, the following statement is true:

Theorem 3. *Let* $\operatorname{rank} Q = n_1 \le \min(k, r)$. *Then the homogeneous* $(f(t) = 0)$ *boundary-value problem (22) corresponding to (3), (4) has exactly* $\rho = r - n_1$ *linearly independent solutions.*

The inhomogeneous boundary-value problem (3), (4) is solvable for those and only those $f(t)$ *and* α *for which the following* $r + d$ *linearly independent conditions are satisfied:*

$$\int_a^b Y_r^*(s)f(s)\,ds = 0,$$

$$P_{N_d(Q^*)}\left\{\alpha - \ell\left[f(\cdot) + \Psi_2(\cdot)\int_a^b \Phi_2^*(s)f(s)\,ds\right]\right\} = 0;$$

in this case, it has the ρ*-parameter family of solutions*

$$x(t) = X_\rho(t)c_\rho + \bar{f}(t) + \bar{\Psi}_2(t)\int_a^b \Phi_2^*(s)f(s)\,ds,$$

where $X_\rho(t) = X_r(t)P_{N_\rho(Q)}$ *is the* $n \times \rho$ *fundamental matrix of the boundary-value problem (3), (4),* $\bar{f}(t) = f(t) - X_r(t)Q^+\ell f(\cdot)$, *and* $\bar{\Psi}_2(t) = \Psi_2(t) - X_r(t)Q^+\ell\Psi_2(\cdot)$.

Let $\operatorname{rank} Q = r$. Then it is necessary that the inequality $k \le r$ be true. In this case, the boundary-value problem (3), (4) is overdetermined, and the following theorem holds for it:

Theorem 4. *Let* $\operatorname{rank} Q = r$. *Then the homogeneous boundary-value problem (22) corresponding to (3), (4) does not have solutions other than the trivial solution.*

The inhomogeneous boundary-value problem (3), (4) is solvable for those and only those $f(t)$ *and* α *that satisfy the following* $r + d$ *linearly independent conditions:*

$$\int_a^b Y_r^*(s) f(s)\, ds = 0,$$

$$P_{N_d(Q^*)}\left\{\alpha - \ell\left[f(\cdot) + \Psi_2(\cdot)\int_a^b \Phi_2^*(s) f(s)\, ds\right]\right\} = 0, \quad d = k - r;$$

in this case, it has the unique solution

$$x(t) = \bar{f}(t) + \bar{\Psi}_2(t)\int_a^b \Phi_2^*(s) f(s)\, ds,$$

where $\bar{f}(t) = f(t) - X_r(t)Q^+\ell f(\cdot)$ *and* $\bar{\Psi}_2(t) = \Psi_2(t) - X_r(t)Q^+\ell\Psi_2(\cdot)$.

Indeed, since $\operatorname{rank} Q = r$, we have $P_{N(Q)} = P_{N_\rho(Q)} = 0$ and $X_\rho(t) = X_r(t)P_{N_\rho(Q)} \equiv 0$, and Theorem 4 follows from Theorem 3.

Let $\operatorname{rank} Q = k$. Then $k \geq r$. In this case, the boundary-value problem (3), (4) is underdetermined, and the following theorem holds for it:

Theorem 5. *Let* $\operatorname{rank} Q = k$. *Then the homogeneous boundary-value problem (22) corresponding to (3), (4) has exactly* $\rho = r - k$ *linearly independent solutions.*

The inhomogeneous boundary-value problem (3), (4) is solvable for those and only those $f(t)$ *that satisfy the following* r *linearly independent conditions:*

$$\int_a^b Y_r^*(s) f(s)\, ds = 0;$$

in this case, it has the ρ*-parameter family of solutions*

$$x(t) = X_\rho(t)c_\rho + \bar{f}(t) + \bar{\Psi}_2(t)\int_a^b \Phi_2^*(s) f(s)\, ds,$$

where $X_\rho(t) = X_r(t)P_{N_\rho(Q)}$ *is an* $n \times \rho$ *fundamental matrix,* $\bar{f}(t) = f(t) - X_r(t)Q^+\ell f(\cdot)$, *and* $\bar{\Psi}_2(t) = \Psi_2(t) - X_r(t)Q^+\ell\Psi_2(\cdot)$.

Since $\operatorname{rank} Q = k$, we have $P_{N(Q^*)} = P_{N_d(Q^*)} = 0$, and Theorem 3 yields Theorem 5.

In the case of the periodic boundary-value problem (3), (4), the following statement is true:

Theorem 6. *Suppose that the elements of the column vector* $f(t)$ *and matrices* $\Phi(t)$ *and* $\Psi(t)$ *are functions periodic in* T *and* $\operatorname{rank} Q = n_1 \leq \min(k, r)$. *Then the homogeneous boundary-value problem corresponding to (3), (4) has exactly* $\rho = r - n_1$ *linearly independent periodic solutions.*

The inhomogeneous boundary-value problem (3), (4) has periodic solutions for those and only those $f(t)$ *that satisfy the following conditions:*

$$\int_0^T Y_r^*(s) f(s)\, ds = 0,$$

$$P_{N_d(Q^*)} \left\{ f(0) - f(T) + [\Psi_2(0) - \Psi_2(T)] \int_0^T \Phi_2^*(s) f(s)\, ds \right\} = 0;$$

in this case, it has the ρ-parameter family of periodic solutions

$$x(t) = X_\rho(t) c_\rho + \overline{f}(t) + \overline{\Psi}_2(t) \int_0^T \Phi_2^*(s) f(s)\, ds,$$

where $X_\rho(t) = X_r(t) P_{N_\rho(Q)}$ *is an* $n \times \rho$ *fundamental matrix,* $\overline{f}(t) = f(t) - X_r(t) Q^+[f(0) - f(T)]$, *and* $\overline{\Psi}_2(t) = \Psi_2(t) - X_r(t) Q^+[\Psi_2(0) - \Psi_2(T)]$.

Example 1. As an illustration of the algorithm proposed above for the construction of solutions of linear boundary-value problems for integral Fredholm equations of the second kind, we consider the boundary-value problem

$$(Lx)(t) = x(t) - \begin{pmatrix} 0 & 0 & t-1 \\ 1 & 0 & 0 \end{pmatrix} \int_0^2 \begin{pmatrix} 0 & s - \frac{1}{2} \\ 1 & 0 \\ \frac{3s}{2} & 0 \end{pmatrix} x(s)\, ds = f(t), \tag{25}$$

$$\ell x(\cdot) = x(0) - x(2) = 0, \tag{26}$$

where $x(t) = \operatorname{col}(x_1(t), x_2(t))$ and $f(t) = \operatorname{col}(f_1(t), f_2(t))$ belong to the space $\mathbf{L}_2^2[0,2]$, $t \in [0,2]$.

The operator L^* adjoint to the operator L has the form

$$(L^* y)(t) = y(t) - \begin{pmatrix} 0 & 1 & \frac{3t}{2} \\ t - \frac{1}{2} & 0 & 0 \end{pmatrix} \int_0^2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ s-1 & 0 \end{pmatrix} y(s)\, ds.$$

Let us determine bases of the kernels $\ker L$ and $\ker L^*$ of the operators L and L^*. To this end, we solve the homogeneous equations

$$\begin{aligned} (Lx)(t) &= 0, \\ (L^* y)(t) &= 0. \end{aligned} \tag{27}$$

We seek solutions in the form

$$x(t) = \begin{pmatrix} 0 & 0 & t-1 \\ 1 & 0 & 0 \end{pmatrix} c,$$

$$y(t) = \begin{pmatrix} 0 & 1 & \frac{3t}{2} \\ t - \frac{1}{2} & 0 & 0 \end{pmatrix} d. \tag{28}$$

Substituting (28) into (27), we obtain the following algebraic systems for $c \in \mathbf{R}^3$ and $d \in \mathbf{R}^3$:

$$(E - A)\, c = 0,$$

$$(E - A^*)\, d = 0, \tag{29}$$

where

$$A = \int_0^2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ t-1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & \frac{3t}{2} \\ t - \frac{1}{2} & 0 & 0 \end{pmatrix} dt = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$D = D^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$P_{N(D)} = P_{N(D^*)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $\operatorname{rank} D = 1$, $r = 2$, and

$$P_{N_r(D)} = P_{N_r(D^*)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix},$$

the solutions of the algebraic equations (29) have the form

$$c = P_{N_r(D)} c_r, \quad c_r \in \mathbf{R}^2,$$

$$d = P_{N_r(D^*)} d_r, \quad d_r \in \mathbf{R}^2.$$

Correspondingly, the general solutions of Eqs. (27) can be written as follows:

$$x(t) = X_r(t)c_r = \begin{pmatrix} 0 & t-1 \\ 1 & 0 \end{pmatrix} c_r,$$

$$y(t) = Y_r(t)d_r = \begin{pmatrix} 0 & \dfrac{3t}{2} \\ t-\dfrac{1}{2} & 0 \end{pmatrix} d_r,$$

where

$$X_r(t) = \begin{pmatrix} 0 & 0 & t-1 \\ 1 & 0 & 0 \end{pmatrix} P_{N_r(D)}, \quad Y_r(t) = \begin{pmatrix} 0 & 1 & \dfrac{3t}{2} \\ t-\dfrac{1}{2} & 0 & 0 \end{pmatrix} P_{N_r(D^*)}.$$

Thus, the columns of the matrices $X_r(t)$ and $Y_r(t)$ form bases of the null spaces $N(L)$ and $N(L^*)$ of the operators L and L^*, respectively.

Using relations (7) and (8), we construct the operators

$$(P_{N(L^*)}y)(t) = \begin{pmatrix} 0 & \dfrac{t}{4} \\ \dfrac{6t-3}{7} & 0 \end{pmatrix} \int_0^2 \begin{pmatrix} 0 & s-\dfrac{1}{2} \\ \dfrac{3s}{2} & 0 \end{pmatrix} y(s)\,ds,$$

$$(\overline{P}_{N(L^*)}x)(t) = \begin{pmatrix} 0 & \dfrac{9t}{4} \\ \dfrac{2t-1}{4} & 0 \end{pmatrix} \int_0^2 \begin{pmatrix} 0 & 1 \\ s-1 & 0 \end{pmatrix} x(s)\,ds,$$

$$(\overline{P}_{N(L)}y)(t) = \begin{pmatrix} 0 & \dfrac{t-1}{6} \\ \dfrac{6}{7} & 0 \end{pmatrix} \int_0^2 \begin{pmatrix} 0 & s-\dfrac{1}{2} \\ \dfrac{3s}{2} & 0 \end{pmatrix} y(s)\,ds.$$

The operator $(\overline{L}x)(t) = (L + \overline{P}_{N(L^*)})x(t)$ has the form

$$(\overline{L}x)(t) = x(t) - \begin{pmatrix} 0 & 0 & t-1 & 0 & \dfrac{9t}{4} \\ 1 & 0 & 0 & \dfrac{-2t+1}{4} & 0 \end{pmatrix} \int_0^2 \begin{pmatrix} 0 & 1 & \dfrac{3s}{2} & 0 & s-1 \\ s-\dfrac{1}{2} & 0 & 0 & 1 & 0 \end{pmatrix}^* x(s)\,ds.$$

Using relation (17), we construct the operator $\overline{L}^{-1}$ inverse to $\overline{L}$:

$$(\overline{L}^{-1}f)(t) = f(t) - \begin{pmatrix} 0 & 0 & t-1 & 0 & \frac{9t}{4} \\ 1 & 0 & 0 & \frac{-2t+1}{4} & 0 \end{pmatrix} M^{-1} \int_0^2 \begin{pmatrix} 0 & 1 & \frac{3s}{2} & 0 & s-1 \\ s-\frac{1}{2} & 0 & 0 & 1 & 0 \end{pmatrix}^* f(s)\,ds,$$

where

$$M^{-1} = \begin{pmatrix} \frac{9}{23} & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{5}{12} & 0 & -\frac{3}{2} \\ \frac{12}{23} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{9} & 0 & 0 \end{pmatrix}.$$

Then, following (18), we rewrite the operator L^+ in the form

$$(L^+f)(t) = ((\overline{L}^{-1} - \overline{P}_{N(L)})f)(t)$$

$$= f(t) - \begin{pmatrix} 0 & 0 & \frac{-2t+5}{12} & 0 & \frac{-3(t-1)}{2} & 0 & \frac{1-t}{6} \\ \frac{-6(t-2)}{23} & 0 & 0 & -\frac{1}{2} & 0 & \frac{-6}{7} & 0 \end{pmatrix}$$

$$\times \int_0^2 \begin{pmatrix} 0 & 1 & \frac{3s}{2} & 0 & s-1 & 0 & \frac{3s}{2} \\ \frac{s-1}{2} & 0 & 0 & 1 & 0 & s-\frac{1}{2} & 0 \end{pmatrix}^* f(s)\,ds. \tag{30}$$

Under the condition

$$(P_{N(L^*)}f)(t) = \begin{pmatrix} 0 & \frac{t}{4} \\ \frac{6t-3}{7} & 0 \end{pmatrix} \int_0^2 \begin{pmatrix} 0 & \frac{s-1}{2} \\ \frac{3s}{2} & 0 \end{pmatrix} f(s)\,ds = 0, \tag{31}$$

the general solution of Eq. (25) has the form

$$x(t) = X_r(t)c_r + (L^+ f)(t) = \begin{pmatrix} 0 & t-1 \\ 1 & 0 \end{pmatrix} c_r + (L^+ f)(t), \quad c_r \in \mathbf{R}^2,$$

where $(L^+ f)(t)$ admits representation (30).

Taking into account that $f(t) = \operatorname{col}(f_1(t), f_2(t))$, we can rewrite condition (31) in the following simpler form:

$$\int_0^2 \begin{pmatrix} 0 & \dfrac{s-1}{2} \\ \dfrac{3s}{2} & 0 \end{pmatrix} \begin{pmatrix} f_1(s) \\ f_2(s) \end{pmatrix} ds = \int_0^2 \begin{pmatrix} \dfrac{s-1}{2} f_2(s) \\ \dfrac{3s}{2} f_1(s) \end{pmatrix} ds = 0.$$

We now establish conditions for the solvability of the boundary-value problem (23), (25) and determine the general form of its solution.

Substituting the general solution of the integral equation (25) into the boundary conditions (26), we obtain the algebraic equation

$$Qc_r = -(L^+ f)(0) + (L^+ f)(2) \tag{32}$$

for the determination of the constant c_r.

For this problem, we have

$$Q = X_r(0) - X_r(2) = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}, \quad Q^+ = \begin{pmatrix} 0 & 0 \\ -\dfrac{1}{2} & 0 \end{pmatrix},$$

$$P_{N(Q)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_{N(Q^*)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $\operatorname{rank} Q = 1$ and $\rho = 1$, we get

$$P_{N_\rho(Q)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad P_{N_\rho(Q^*)} = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

Equation (32) is solvable under the condition

$$P_{N_\rho(Q^*)}\{f(0) - f(2) + (L^+ f)(0) - (L^+ f)(2)\} = 0,$$

which, with regard for the fact that

$$P_{N_\rho(Q^*)} = \begin{pmatrix} 0 & 1 \end{pmatrix} \qquad \text{and} \qquad f(t) = \operatorname{col}(f_1(t), f_2(t)),$$

takes the form

$$(0 \quad 1)\left\{\begin{pmatrix} f_1(0) \\ f_2(0) \end{pmatrix} - \begin{pmatrix} f_1(2) \\ f_2(2) \end{pmatrix} + \begin{pmatrix} 0 & 0 & \frac{1}{3} & 0 & 3 & 0 & 0 \\ \frac{12}{23} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right.$$

$$\left. \times \int_0^2 \begin{pmatrix} 0 & 1 & \frac{3s}{2} & 0 & s-1 & 0 & \frac{3s}{2} \\ \frac{s-1}{2} & 0 & 0 & 1 & 0 & \frac{-1}{2} & 0 \end{pmatrix}^* \begin{pmatrix} f_1(s) \\ f_2(s) \end{pmatrix} ds \right\} = 0.$$

After transformations, we obtain

$$f_2(0) - f_2(2) + \frac{6}{23}\int_0^2 (s-1) f_2(s)\, ds. \tag{33}$$

Under condition (33), Eq. (32) has solutions of the form

$$c_r = \begin{pmatrix} 1 \\ 0 \end{pmatrix} c_\rho - Q^+ \left\{ f(0) - f(2) + \begin{pmatrix} 0 & 0 & \frac{1}{3} & 0 & 3 & 0 & 0 \\ \frac{12}{23} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right.$$

$$\left. \times \int_0^2 \begin{pmatrix} 0 & 1 & \frac{3s}{2} & 0 & s-1 & 0 & \frac{3s}{2} \\ \frac{s-1}{2} & 0 & 0 & 1 & 0 & s-\frac{1}{2} & 0 \end{pmatrix}^* f(s)\, ds \right\}, \quad c_\rho \in \mathbf{R}^1.$$

Thus, problem (25), (26) is solvable under conditions (31) and (33) and has the following general solution:

$$x(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} c_\rho + \begin{pmatrix} \frac{1-t}{2} & 0 \\ 0 & 0 \end{pmatrix} \{f(0) - f(2)\} + f(t)$$

$$+ \begin{pmatrix} 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1-t}{6} \\ \frac{6(2-t)}{23} & 0 & 0 & 0 & \frac{3(1-t)}{2} & \frac{-6}{7} & 0 \end{pmatrix}$$

$$\times \int_0^2 \begin{pmatrix} 0 & 1 & \frac{3s}{2} & 0 & s-1 & 0 & \frac{3s}{2} \\ \frac{s-1}{2} & 0 & 0 & 1 & 0 & s-\frac{1}{2} & 0 \end{pmatrix}^* f(s)\, ds.$$

Example 2. Consider the boundary-value problem for Eq. (25) with the boundary conditions

$$\ell x(\cdot) = \int_0^2 \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{3t}{2} & 0 \\ 0 & \frac{t}{2} \end{pmatrix} x(t)\,dt = \alpha. \tag{34}$$

The boundary-value problem (25), (34) is overdetermined. We seek its solution using Theorem 4.

Substituting the solution of Eq. (25) into the boundary conditions (34), we obtain the following algebraic equation for c_r:

$$Qc_r = \alpha - \ell L^+ f(\cdot)$$

$$= \alpha - \int_0^2 \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{3s}{2} & 0 \\ 0 & \frac{s}{2} \end{pmatrix} f(s)\,ds - \Psi_2 \int_0^2 \begin{pmatrix} 0 & 1 & \frac{3s}{2} & 0 & s-1 & 0 & \frac{3s}{2} \\ \frac{s-1}{2} & 0 & 0 & 1 & 0 & s-\frac{1}{2} & 0 \end{pmatrix}^* f(s)\,ds, \tag{35}$$

where

$$Q = \ell X_r(\cdot) = \int_0^2 \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{3t}{2} & 0 \\ 0 & \frac{t}{2} \end{pmatrix} \begin{pmatrix} 0 & t-1 \\ 1 & 0 \end{pmatrix} dt,$$

$$\Psi_2 = \ell\Psi_2(\cdot) = \int_0^2 \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{3t}{2} & 0 \\ 0 & \frac{t}{2} \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{-2t+5}{12} & 0 & \frac{-3(t-1)}{2} & 0 & \frac{1-t}{6} \\ \frac{-6(t-2)}{23} & 0 & 0 & -\frac{1}{2} & 0 & \frac{-6}{7} & 0 \end{pmatrix} dt$$

$$= \begin{pmatrix} \frac{6}{23} & 0 & 0 & -\frac{1}{2} & 0 & -\frac{6}{7} & 0 \\ 0 & 0 & \frac{7}{12} & 0 & -\frac{3}{2} & 0 & -\frac{2}{3} \\ -\frac{4}{23} & 0 & 0 & -\frac{1}{2} & 0 & -\frac{6}{7} & 0 \end{pmatrix}.$$

For this boundary-value problem, we have

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Q^+ = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix},$$

$$P_{N(Q)} = 0, \quad P_{N(Q^*)} = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Since $\operatorname{rank} Q = 2$ and $\rho = 2$, we have

$$P_{N_\rho(Q)} = 0 \quad \text{and} \quad P_{N_d(Q^*)} = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}.$$

Equation (35) is solvable under the condition

$$P_{N_d(Q^*)} \left\{ \alpha - \ell \left[f(\cdot) + \Psi_2(\cdot) \int_0^2 \Phi_2^*(s) f(s)\, ds \right] \right\} = 0,$$

which, with regard for the fact that

$$P_{N_d(Q^*)} = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}, \quad f(t) = \operatorname{col}(f_1(t), f_2(t)), \quad \text{and} \quad \alpha = \operatorname{col}(\alpha_1, \alpha_2, \alpha_3),$$

takes the form

$$\begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \left\{ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} - \int_0^2 \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{3s}{2} & 0 \\ 0 & \frac{s}{2} \end{pmatrix} \begin{pmatrix} f_1(s) \\ f_2(s) \end{pmatrix} ds \right.$$

$$- \begin{pmatrix} \frac{6}{23} & 0 & 0 & -\frac{1}{2} & 0 & -\frac{6}{7} & 0 \\ 0 & 0 & \frac{7}{12} & 0 & -\frac{3}{2} & 0 & -\frac{2}{3} \\ -\frac{4}{23} & 0 & 0 & -\frac{1}{2} & 0 & -\frac{6}{7} & 0 \end{pmatrix}$$

$$\left. \times \int_0^2 \begin{pmatrix} 0 & 1 & \frac{3s}{2} & 0 & s-1 & 0 & \frac{3s}{2} \\ \frac{s-1}{2} & 0 & 0 & 1 & 0 & s-\frac{1}{2} & 0 \end{pmatrix}^* \begin{pmatrix} f_1(s) \\ f_2(s) \end{pmatrix} ds \right\} = 0.$$

After transformations, we get

$$\alpha_1 - \alpha_3 + \frac{13}{46}\int_0^2 (s-1) f_2(s)\, ds = 0. \tag{36}$$

Under condition (36), Eq. (35) has a unique solution of the form

$$c_r = Q^+ \left\{ \alpha - \int_0^2 \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{3s}{2} & 0 \\ 0 & \frac{s}{2} \end{pmatrix} f(s)\, ds - \Psi_2 \int_0^2 \begin{pmatrix} 0 & 1 & \frac{3s}{2} & 0 & s-1 & 0 & \frac{3s}{2} \\ \frac{s-1}{2} & 0 & 0 & 1 & 0 & s-\frac{1}{2} & 0 \end{pmatrix}^* f(s) ds \right\}.$$

Thus, the boundary-value problem (25), (34) is solvable under conditions (31) and (36) and has a unique solution of the form

$$x(t) = \begin{pmatrix} 0 & t-1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \left\{ \alpha - \int_0^2 \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{3s}{2} & 0 \\ 0 & \frac{s}{2} \end{pmatrix} f(s)\, ds \right.$$

$$\left. - \Psi_2 \int_0^2 \begin{pmatrix} 0 & 1 & \frac{3s}{2} & 0 & s-1 & 0 & \frac{3s}{2} \\ \frac{s-1}{2} & 0 & 0 & 1 & 0 & s-\frac{1}{2} & 0 \end{pmatrix}^* f(s)\, ds \right\} + (L^+ f)(t).$$

Example 3. Consider the boundary-value problem for Eq. (25) with the boundary conditions

$$\ell x(\cdot) = \begin{pmatrix} -1 & 2 \end{pmatrix} x(0) + \begin{pmatrix} 1 & -2 \end{pmatrix} x(2) = \alpha, \quad \alpha \in \mathbf{R}^1. \tag{37}$$

The boundary-value problem (25), (37) is underdetermined. We seek its solution using Theorem 5.

Substituting the solution of Eq. (25) into the boundary conditions (37), we obtain the following algebraic equation for c_r:

$$Qc_r = \alpha - \ell L^+ f(\cdot) = \alpha - \overline{f} - \overline{\Psi}_2 \int_0^2 \begin{pmatrix} 0 & 1 & \frac{3s}{2} & 0 & s-1 & 0 & \frac{3s}{2} \\ \frac{s-1}{2} & 0 & 0 & 1 & 0 & s-\frac{1}{2} & 0 \end{pmatrix}^* f(s)\, ds, \tag{38}$$

where

$$Q = \ell X_r(\cdot) = \begin{pmatrix} -1 & 2 \end{pmatrix} X_r(0) + \begin{pmatrix} 1 & -2 \end{pmatrix} X_r(2) = \begin{pmatrix} 0 & 2 \end{pmatrix},$$

$$\bar{f} = \begin{pmatrix} -1 & 2 \end{pmatrix} f(0) + \begin{pmatrix} 1 & -2 \end{pmatrix} f(2),$$

$$\bar{\Psi}_2 = \ell\Psi_2(\cdot) = \begin{pmatrix} -1 & 2 \end{pmatrix} \Psi(0) + \begin{pmatrix} 1 & -2 \end{pmatrix} \Psi(2) = \begin{pmatrix} \frac{24}{23} & 0 & \frac{1}{3} & 0 & -3 & -\frac{24}{7} & -\frac{1}{3} \end{pmatrix}.$$

For this boundary-value problem, we have

$$Q^+ = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \quad P_{N(Q)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_{N(Q^*)} = 0.$$

Since $\operatorname{rank} Q = 1$, $\rho = 1$, and $d = 0$, we get

$$P_{N_\rho(Q)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad P_{N_d(Q^*)} = 0.$$

Since $P_{N_d(Q^*)} = 0$, Eq. (38) is solvable for any right-hand side and has a one-parameter $(c_\rho \in \mathbf{R}^1)$ family of solutions of the form

$$c_r = \begin{pmatrix} 1 \\ 0 \end{pmatrix} c_\rho + Q^+ \left\{ \alpha - \bar{f} - \bar{\Psi}_2 \int_0^2 \begin{pmatrix} 0 & 1 & \frac{3s}{2} & 0 & s-1 & 0 & \frac{3s}{2} \\ \frac{s-1}{2} & 0 & 0 & 1 & 0 & s-\frac{1}{2} & 0 \end{pmatrix}^* f(s)ds \right\}.$$

Thus, the boundary-value problem (25), (37) is solvable under condition (31) and has a general solution of the form

$$x(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} c_\rho + \begin{pmatrix} \frac{t-1}{2} \\ 0 \end{pmatrix}$$

$$\times \left\{ \alpha - \bar{f} - \bar{\Psi}_2 \int_0^2 \begin{pmatrix} 0 & 1 & \frac{3s}{2} & 0 & s-1 & 0 & \frac{3s}{2} \\ \frac{s-1}{2} & 0 & 0 & 1 & 0 & s-\frac{1}{2} & 0 \end{pmatrix}^* f(s)\, ds \right\} + (L^+ f)(t).$$

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