PERIODIC SOLUTIONS OF NONLINEAR AUTONOMOUS SYSTEMS IN CRITICAL CASES

A. A. Boichuk, V. F. Zhuravlev, and S. M. Chuiko

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We obtain coefficient conditions of existence and an iterational algorithm for constructing periodic solutions of weakly perturbed autonomous nonlinear differential systems in critical cases in the presence of multiple roots for generating amplitudes.

1. Statement of the Problem and Notation. We consider the problem of finding coefficient conditions of existence of periodic solutions and their construction for weakly-nonlinear autonomous systems:

$$dz/dt = Az + \varepsilon Z(z, \varepsilon). \tag{1}$$

Here we investigate the critical case in which the $(n \times n)$-dimensional matrix A has characteristic numbers on the imaginary axis of the form $ik2\pi/T$, $k = 0, \pm 1, \ldots$; $i = \sqrt{-1}$; $Z(z, \varepsilon)$ is an n-dimensional vector-valued function belonging to the class of functions continuously differentiable in the first argument and continuous in the second, $(Z(\cdot, \varepsilon) \in C^1 [\|z - z_0\| \leqslant q]$, $Z(z, \cdot) \in C[0, \varepsilon_0])$, in a neighborhood of generating solutions. A generating system, obtained from Eq. (1) for $\varepsilon = 0$,

$$dz/dt = Az \tag{2}$$

has an m-parameter family of generating T-periodic solutions of the form

$$z(t, c) = X_m(t)c, \quad c \in R^m, \tag{3}$$

where $X_m(t)$ is an $(n \times m)$-dimensional matrix whose columns comprise a complete system of m linearly independent T-periodic solutions of system (2).

We find conditions of existence of periodic solutions of system (1), which for $\varepsilon = 0$ revert into one of the generating solutions (3) of system (2).

As is well known [1-3], the problem of finding periodic solutions of autonomous systems differs in an essential way from the analogous problems for nonautonomous systems in that, in contrast to the latter, a period of a desired solution of system (1) is unknown and depends on the parameter ε: $T_1(e) = T(1 + \varepsilon\alpha(\varepsilon))$. The quantity $\alpha = \alpha(\varepsilon)$, $\alpha(0) = \alpha^*$ is subject to definition in the process of finding the solution itself. Moreover, in finding a periodic solution of period $T_1(\varepsilon)$ of system (1) that reverts to a generative T-periodic solution (3) for $\varepsilon = 0$, we can, with no loss of generability, assume [2] the last component of the m-dimensional column $s \in R^m$ to be zero, so that the generative solution (3) will depend on the $(m-1)$-th constant $\bar{c} \in R^{m-1}$; $c = \mathrm{col}\,(\bar{c}, 0) \in R^m$.

Making a change in the independent variable $t = \tau(1 + \varepsilon\alpha)$, we reduce problem (1) to the problem of finding a T-periodic solution $z(\tau, \varepsilon)$ of the system

$$\dot{z} = Az + \varepsilon\{\alpha Az + (1 + \varepsilon\alpha) Z(z, \varepsilon)\} \quad (dz/d\tau = \dot{z}), \tag{4}$$

which reverts for $\varepsilon = 0$ into one of the generating T-periodic solutions

$$z_0(\tau, \bar{c}) = X_{m-1}(\tau)\bar{c}, \quad \bar{c} \in R^{m-1}, \tag{5}$$

where $X_{m-1}(\tau)$ is an $[n \times (m-1)]$-dimensional matrix analogous to the matrix $X_m(\tau)$ minus its m-th column.

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2. Necessary Condition of Existence. By the traditional change of independent variable

$$z(\tau,\varepsilon) = z_0(\tau,\bar{c}) + x(\tau,\varepsilon) \tag{6}$$

we reduce problem (4) to the problem of finding a T-periodic solution $x(\tau,\varepsilon) \in C[\varepsilon]$ of the system

$$\dot{x} = Ax + \varepsilon\{\alpha A(z_0 + x) + (1+\varepsilon\alpha) Z(z_0 + x, \varepsilon)\}, \tag{7}$$

which reverts for $\varepsilon = 0$ to the null solution $x(\tau, 0) = 0$.

It is easy to establish [2] a necessary condition of existence of a T-periodic solution of system (7) and, by the same token, also a $T_1(\varepsilon)$-periodic solution of system (1).

THEOREM 1. Let the autonomous differential system (1) have a periodic solution $z(t, \varepsilon)$ with period $T_1(\varepsilon) = T(1 + \varepsilon\alpha)$, which reverts for $\varepsilon = 0$ to a generating T-periodic solution $z_0(t, \bar{c}^*)$ (5) with constant $\bar{c}^* \in R^{m-1}$, where $\alpha(0) = \alpha^* \in R^1$. Then the vector constant $c^* = \mathrm{col}(\bar{c}^*, \alpha^*) \in R^m$ satisfies the equation

$$F(c) = \int_0^T H(s)\{\alpha A z_0(s,\bar{c}) + Z(z_0(s,\bar{c}), 0)\}\, ds = 0. \tag{8}$$

By analogy with the case of the periodic problem for a nonautonomous system [2, 4], we refer to Eq. (8) as an equation for generting amplitudes of the problem for periodic solutions of the autonomous system (1). $H(t)$ is an $(m \times n)$-matrix whose rows constitute a complete system of m linearly indpendent T-periodic solutions of the system conjugate to system (2).

As in the nonautonomous case, Eq. (8) determines the amplitude of a generating solution, i.e., the $(m - 1)$-dimensional vector constant $\bar{c}^*$. In addition, Eq. (8) yields the scalar constant α^*, characterizing the first correction to the period of the sought-for solution. As a result, just as in the nonautonomous case, Eq. (8) consists of m equations (algebraic or transcendental) for the m unknowns $c = \mathrm{col}(\bar{c}, \alpha) \in R^m$.

Since the analysis takes us into the real domain, the discussion centers on the real roots of Eq. (8).

3. Sufficient Conditions of Existence. Simple Roots of the Equation for Generating Amplitudes. We determine sufficient conditions for the existence of periodic solutions $z(t,\varepsilon) \in C[\varepsilon]$, $z(t, 0) = z_0(t, \bar{c}^*)$ with period $T_1(\varepsilon) = T(1 + \varepsilon\alpha)$, $\alpha(0) = \alpha^*$ of the autonomous system (1), where $c^* = \mathrm{col}(\bar{c}^*, \alpha^*)$ satisfy Eq. (8) for generating amplitudes. By virtue of relations (3) and (6), we shall solve the problem of finding conditions for the existence of T-periodic solutions $x(\tau,\varepsilon) \in C[\varepsilon]$, $x(\tau, 0) = 0$ of system (7).

We assume that c^* is a simple root of Eq. (8). The condition $\det[\partial F(c)/\partial c] \neq 0$ for simplicity fo a root $c = c^*$ of the equation for generating amplitude is equivalent to the condition

$$\det B_0 \neq 0, \tag{9}$$

where $B_0 = \int_0^T H(s)\bar{A}_1(s)\, ds$ is an $(m \times m)$-dimensional matrix, $\bar{A}_1(s) = [(\alpha^* A + A_1(s)) X_{m-1}(s), A X_{m-1}(s)\bar{c}^*]$ is an $(n \times m)$-dimensional matrix, and $A_1(s) = \left.\frac{\partial Z(z,0)}{\partial z}\right|_{z=z_0(s\bar{c}^*)}$ is an $(n \times n)$-dimensional matrix.

Indeed, from Eq. (8) we have

$$\left.\frac{\partial F(c)}{\partial c}\right|_{c=c^*} = \int_0^T H(s)[\alpha^* A X_{m-1}(s), A X_{m-1}(s)\bar{c}^*]\, ds + \int_0^T H(s)[A_1(s) X_{m-1}(s), 0]\, ds = B_0,$$

where the square brackets contain block matrices of dimensions $n \times (m - 1)$ and $n \times 1$, respectively.

In system (7), considering the expression in the braces as a non-homogeneity, we proceed, by analogy with [4, 5], from a periodic boundary value problem for system (7) to the following equivalent operator system on the set $x(\tau,\varepsilon) \in C[\varepsilon]$, $x(\tau, 0) = 0$:

$$x(\tau,\varepsilon) = X_{m-1}(\tau)\bar{c} + x^{(1)}(\tau,\varepsilon),$$

$$\int_0^T H(s)\{\alpha A(z_0(s,\bar{c}^*) + x(s,\varepsilon)) + (1+\varepsilon\alpha) Z(z_0 + x, \varepsilon)\}\, ds = 0,$$

$$x^{(1)}(\tau,\varepsilon)=\varepsilon\int_0^T G(\tau,s)\{\alpha A(z_0+x)+(1+\varepsilon\alpha)Z(z_0+x,\varepsilon)\}\,ds,$$

where $G(t,\tau)$ is the generalized Green's matrix of the problem concerning T-periodic solutions of system (2) [5, 6].

Separating out from the vector-valued function $Z(z_0+x,\varepsilon)$ the linear part in x and the zero-order terms in ε, we obtain

$$Z(z_0+x,\varepsilon)=Z(z_0,0)+A_1(\tau)x+\varphi(x,\varepsilon), \tag{10}$$

where $\varphi(0,0)=0,\ \partial\varphi(0,0)/\partial x=0$.

We therefore have the following representation for the expression in braces $\{\ldots\}$:

$$\begin{aligned}\{\ldots\}&=\alpha A(z_0+x)+(1+\varepsilon\alpha)Z(z_0+x,\varepsilon)=\alpha^*Az_0+Z(z_0,0)+\\&+(\alpha^*A+A_1(\tau))x+\bar\alpha Az_0+\bar\alpha Ax+\varphi(x,\varepsilon)+\varepsilon\alpha Z(z_0+x,\varepsilon)=\\&=f_0(\tau,c^*)+\bar A_1(\tau)c+(\alpha^*A+A_1(\tau))x^{(1)}(\tau,\varepsilon)+R(x(\tau,\varepsilon),\varepsilon),\end{aligned} \tag{11}$$

where

$$f_0(\tau,c^*)=\alpha^*Az_0(\tau,\bar c^*)+Z(z_0(\tau,\bar c^*),0),$$

$$R(x,\varepsilon)=\bar\alpha Ax+\varphi(x,\varepsilon)+\varepsilon\alpha Z(z_0+x,\varepsilon),$$

$$\bar\alpha=\alpha-\alpha^*\quad R(0,0)=0,\quad \partial R(0,0)/\partial x=0.$$

Taking into account the fact that the constant $c^*\in R^m$ satisfies Eq. (8) for generating amplitudes, we obtain the following equivalent operator system for the determination of a T-periodic solution $x(\tau,\varepsilon)\in C[\varepsilon],\ x(\tau,0)=0$ of the boundary value problem (7):

$$\begin{aligned}&x(\tau,\varepsilon)=X_{m-1}(\tau)I_1c+x^{(1)}(\tau,\varepsilon),\\&B_0c=-\int_0^T H(s)\{(\alpha^*A+A_1(s))x^{(1)}(s,\varepsilon)+R(x,\varepsilon)\}\,ds,\\&x^{(1)}(\tau,\varepsilon)=\varepsilon\int_0^T G(\tau,s)\{f_0(s,c^*)+\bar A_1(s)c+(\alpha^*A+A_1(s))x^{(1)}(s,\varepsilon)+R(x,\varepsilon\}\,ds,\end{aligned} \tag{12}$$

where $I_1=\begin{bmatrix}1&0&\ldots&0&0\\0&1&\ldots&0&0\\ \cdot&\cdot&\cdot&\cdot&\cdot\\0&0&\ldots&1&0\end{bmatrix}$ is an $[(m-1)\times m]$-dimensional matrix.

The operator system (12) belongs to the class of systems of the form (5.11) in [4] for whose solution we apply the method of simple iterations converging on $[0,\varepsilon_0]$. Thus we have the following theorem.

THEOREM 2. Assume that system (4) satisfies the conditions indicated above. Then for each simple ($\det B_0\neq 0$) root $c^*=\mathrm{col}(\bar c^*,\alpha^*)\in R^m$ of Eq. (8) for generating amplitudes the system (4) has a unique T-period solution $z(\tau,\varepsilon)\in C[\varepsilon]$, reverting for $\varepsilon=0$ into the generating solution $z_0(\tau,\bar c^*)$ of (5). This solution can be determined with the aid of the following iterational process converging on $[0,\varepsilon_0]$:

$$\begin{aligned}&c_k=-B_0^{-1}\int_0^T H(s)\{(\alpha^*A+A_1(s))x_k^{(1)}(s,\varepsilon)+R(x_k(s,\varepsilon),\varepsilon)\}\,ds,\\&x_{k+1}^{(1)}(\tau,\varepsilon)=\varepsilon\int_0^T G(\tau,s)\{f_0(s,c^*)+\bar A_1(s)c_k+(\alpha^*A+A_1(s))x_k^{(1)}(s,\varepsilon)+R(x_k,\varepsilon)\}\,ds,\\&x_{k+1}(\tau,\varepsilon)=X_{m-1}(\tau)I_1c_k+x_{k+1}^{(1)}(\tau,\varepsilon),\\&z_{k+1}(\tau,\varepsilon)=z_0(\tau,\bar c^*)+x_{k+1}(\tau,\varepsilon),\quad k=0,1,2,\ldots;\quad x_0(\tau,\varepsilon)=x_0^{(1)}(\tau,\varepsilon)=0.\end{aligned} \tag{13}$$

With a change of independent variable taken into account, the iterational scheme (13) yields a periodic solution with period $T_1(\varepsilon)=T(1+\varepsilon\alpha)$, $\alpha(0)=\alpha^*$ of the autonomous system (1). Moreover, the m-th component of the vector constant $c_k=c_k(\varepsilon)=\mathrm{col}(\bar c_k,\alpha_k)\in R^m$ will yield approximations $\alpha_k=\alpha_k(\varepsilon)$ to $\alpha(\varepsilon)\in R^1$.

As in the case of nonautonomous systems [4], we refer to the case $\det B_0 \neq 0$ as a critical case of the first order. It is distinguished by the fact that it furnishes an answer to the problem of the existence of a T-periodic solution of system (4), following an analysis of the system of equations used for finding a first approximation $x_1(\tau, \varepsilon)$ to the desired solution

4. Multiple Roots of the Equation for Generating Amplitudes. We assume now that $c = c^*$ is a multiple root of Eq. (8) for generating amplitudes, i.e., $\det B_0 = 0$. Let P_0 and $P_0^{(*)}$ denote orthoprojectors onto the origin of the spaces $N(B_0)$ and $N(B_0^*)$ of matrices B_0 and $B_0^* = B_0^T$, respectively. The second equation in system (12) is then solvable if and only if

$$P_0^{(*)} \int_0^T H(s)\{(\alpha^* A + A_1(s))\, x^{(1)}(s, \varepsilon) + R(x(s, \varepsilon), \varepsilon)\}\, ds = 0, \tag{14}$$

and then has the solution

$$c = -B_0^+ \int_0^T H(s)\{(\alpha^* A + A_1(s))\, x^{(1)}(s, \varepsilon) + R(x, \varepsilon)\}\, ds + c^{(1)} = (I - P_0)\, c^{(0)} + P_0 c^{(1)},$$

where B_0 is the unique pseudo-inverse matrix to B_0 [7];

$$c^{(0)} = (I - P_0)\, c = (I - P_0)\, c^{(0)} \in R^m \ominus N(B_0), \quad c^{(1)} = P_0 c = P_0 c^{(1)} \in N(B_0). \tag{15}$$

Substituting relation (15) into the third equation of system (12), we obtain

$$x^{(1)}(\tau, \varepsilon) = \varepsilon G_1(\tau)\, P_0 c^{(1)} + x^{(2)}(\tau, \varepsilon), \quad G_1(\tau) = \int_0^T G(\tau, s)\, \bar{A}_1(s)\, P_0\, ds,$$
$$x^{(2)}(\tau, \varepsilon) = \varepsilon \int_0^T G(\tau, s)\{f_0(s, c^*) + \bar{A}_1(s)(I - P_0)\, c^{(0)} +$$
$$+ (\alpha^* A + A_1(s))(\varepsilon G_1(s)\, P_0 c^{(1)} + x^{(2)}(s, \varepsilon)) + R(x(s, \varepsilon), \varepsilon)\}\, ds. \tag{16}$$

To obtain an equivalent operator system of type (12) for the case $\det B_0 = 0$ we need to revise expansion (11) by writing out explicitly the terms linear in x and in ε. To do this we require the additional condition of continuous differentiability with respect to ε of the vector-valued function $Z(z_0 + x, \varepsilon)$, which we assume: $Z(z, \cdot) \in C^1[\varepsilon]$, $\varepsilon \in [0, \varepsilon_0]$. We then write relation (10) in the form

$$Z(z_0 + x, \varepsilon) = Z(z_0, 0) + A_1(\tau)\, x + \varepsilon A_2(\tau)\, x + \varphi_1(x, \varepsilon),$$

where $\varphi_1(0, 0) = 0$, $\partial\varphi_1(0, 0)/\partial x = 0$, $\partial^2\varphi_1(0, 0)/\partial x \partial\varepsilon = 0$, and $A_2(\tau) = \left.\dfrac{\partial^2 Z(z, \varepsilon)}{\partial\varepsilon\partial z}\right|_{\substack{z = z_0(\tau, c^*) \\ \varepsilon = 0}}$ is an $(n \times n)$-dimensional matrix.

Expansion (11) then takes the form

$$\{\ldots\} = f_0(\tau, c^*) + \bar{A}_1(\tau)\, c + (\alpha^* A + A_1(\tau))\, x^{(1)}(\tau, \varepsilon) + \bar{\alpha} A x + \varepsilon A_2(\tau)\, x +$$
$$+ \varphi_1(x, \varepsilon) + \varepsilon\alpha^* Z(z_0, 0) + \varepsilon\alpha^* A_1(\tau)\, x + \varepsilon^2\alpha^* A_2(\tau, x + \varepsilon\alpha^*\varphi_1(x, \varepsilon) +$$
$$+ \varepsilon\bar{\alpha} Z(z_0, 0) + \varepsilon\bar{\alpha} A_1(\tau)\, x + \varepsilon\bar{\alpha}\varphi(x, \varepsilon) = f_0(\tau, c^*) + \bar{A}_1(\tau)\, c +$$
$$+ (\alpha^* A + A_1(\tau))\, x^{(1)}(\tau, \varepsilon) + \varepsilon(\alpha^* A_1(\tau) + A_2(\tau))(X_{m-1}(\tau)\, \bar{c} + x^{(1)}(\tau, \varepsilon)) +$$
$$+ \varepsilon\bar{\alpha} Z(z_0, 0) + \ldots = f_0(\tau, c^*) + \bar{A}_1(\tau) c + (\alpha^* A + A_1(\tau))\, x^{(1)}(\tau, \varepsilon) + \varepsilon\bar{A}_2(\tau)\, c + R_1(x, \varepsilon), \tag{11'}$$

where $\bar{A}_2(\tau) = [(\alpha^* A_1(\tau) + A_2(\tau))\, X_{m-1}(\tau), Z(z_0, 0)]$ is an $(n \times m)$-dimensional matrix, and

$$R_1(x, \varepsilon) = \bar{\alpha} A x + \varphi_1(x, \varepsilon) + \varepsilon\alpha^* Z(z_0, 0) + \varepsilon(\alpha^* A_1(\tau) + A_2(\tau))\, x^{(1)}(\tau, \varepsilon +$$
$$+ \varepsilon^2\alpha^* A_2(\tau)\, x + \varepsilon\alpha^*\varphi_1(x, \varepsilon) + \varepsilon\bar{\alpha} A_1(\tau) + \varepsilon\bar{\alpha}\varphi(x, \varepsilon);$$
$$R_1(0, 0) = 0, \quad \partial R_1(0, 0)/\partial x = 0.$$

Upon taking expressions (15), (16), and (11') into account, we obtain from relation (14) an equation for determination of the unknown $c^{(1)} \in N(B_0) \subset R^m$:

$$\varepsilon B_1 c^{(1)} = -P_0^{(*)} \int_0^T H(s)\{(\alpha^* A + A_1(s))\, x^{(2)}(s, \varepsilon) + \varepsilon\bar{A}_2(s)(I - P_0)\, c^{(0)} + R_1(x, \varepsilon)\}\, ds, \tag{17}$$

where $B_1 = P_0^{(*)} \int_0^T H(s)[(\alpha^* A + A_1(s)) G_1(s) + \bar{A}_2(s)] P_0 ds$ is an (m × m)-dimensional matrix.

Let P_1 and $P_1^{(*)}$ be orthoprojectors onto zero of the spaces $N(B_1)$ and $N(B_1^*)$ of matrices B_1 and $B_1^* = B_1^T$. Then, as in the case of the nonautonomous system [3, 4], providing $P_1^{(*)} P_0^{(*)}$ 0, Eq. (17) is uniquely ($P_0 P_1 = 0$) solvable for $\varepsilon c^{(1)}$:

$$\varepsilon c^{(1)} = -B_1^+ \int_0^T P_0^{(*)} H(s) \{(\alpha^* A + A_1(s)) x^{(2)}(s, \varepsilon) + \varepsilon \bar{A}_2(s)(I - P_0) c^{(0)} + R_1(x, \varepsilon)\} ds,$$

where B_1^+ is a matrix pseudo-inverse to B_1.

Thus, for the case $\det B_0 = 0 \leftrightarrow P_0 \neq 0$, subject to the condition

$$P_1^{(*)} P_0^{(*)} = 0 \tag{18}$$

we proceed from the operator system (12) to the following:

$$\begin{aligned} x(\tau, \varepsilon) &= X_{m-1}(\tau) I_1 (I - P_0) c^{(0)} + \varepsilon G_1(\tau) P_0 c^{(1)} + x^{(2)}(\tau, \varepsilon), \\ c^{(0)} &= -B_0^+ \int_0^T H(s) \{(\alpha^* A + A_1(s))(\varepsilon G_1(s) P_0 c^{(1)} + x^{(2)}(s, \varepsilon)) + R(x, \varepsilon)\} ds, \\ \varepsilon c^{(1)} &= -B_1^+ \int_0^T P_0^{(*)} H(s) \{(\alpha^* A + A_1(s)) x^{(2)}(s, \varepsilon) + \varepsilon \bar{A}_2(s)(I - P_0) c^{(0)} + R_1(x, \varepsilon)\} ds, \\ x^{(2)}(\tau, \varepsilon) &= \varepsilon \int_0^T G(\tau, s) \{f_0(s, c^*) + \bar{A}_1(s)(I - P_0) c^{(0)} + (\alpha^* A + A_1(s))(\varepsilon G_1(s) P_0 c^{(1)} + x^{(2)}) + R(x, \varepsilon)\} ds. \end{aligned} \tag{19}$$

The operator system (19) belongs to the class of systems [5, 6] for whose solutions we apply the convergent method of simple iterations. To find a solution of system (19) and, hence also, to solve system (4), we set up, by analogy with periodic nonautonomous systems [5], an iterational process chaacterized by the fact that the answer to the problem concerning existence of a periodic solution of system (4) is completely determined upon analyzing a system of second approximation to system (4). We shall refer to this case $P_0 \neq 0$, $P_1^{(*)} P_0^{(*)} = 0$ as a critical case of the second order. Thus we have the following theorem.

THEOREM 3. Assume that system (4) satisfies the conditions indicated above, so that

$$P_0 \neq 0, \quad P_1^{(*)} P_0^{(*)} = 0. \tag{20}$$

Then for each root $c = c^* = \mathrm{col}(\bar{c}^*, \alpha^*) \in R^m$ of Eq. (8) for generating amplitudes, the system (4) has, providing

$$\begin{aligned} &\int_0^T P_0^{(*)} H(s) \{(\alpha^* A + A_1(s)) x_1(s, \varepsilon) + R(x_1(s, \varepsilon), \varepsilon)\} ds = 0, \\ &(x_1(\tau, \varepsilon) = \varepsilon \int_0^T G(\tau, s) f_0(s, c^*) ds) \end{aligned} \tag{21}$$

a unique T-periodic solution $z(\tau, \varepsilon) \in C[\varepsilon]$, reverting for $\varepsilon = 0$ to a generating solution $z_0(\tau, \bar{c}^*)$ of the solutions (5). This solution is determined from the operator system (19) with the aid of the following iterational process, converging on $[0, \varepsilon_0]$:

$$\begin{aligned} \varepsilon c_k^{(1)} &= -B_1^+ \int_0^T P_0^{(*)} H(s) \{(\alpha^* A + A_1(s)) x_k^{(2)}(s, \varepsilon) + \varepsilon \bar{A}_2(s)(I - P_0) c_k^{(0)} + R_1(x_k, \varepsilon)\} ds, \\ c_k^{(0)} &= -B_0^+ \int_0^T H(s) \{(\alpha^* A + A_1(s))(\varepsilon G_1(s) P_0 c_k^{(1)} + x_k^{(2)}) + R(x_k, \varepsilon)\} ds, \\ x_{k+1}^{(2)}(\tau, \varepsilon) &= \varepsilon \int_0^T G(\tau, s) \{f_0(s, c^*) + \bar{A}_1(s)(I - P_0) c_k^{(0)} + (\alpha^* A + A_1(s))(\varepsilon G_1(s) P_0 c_k^{(1)} + x_k^{(2)}) + R(x_k \varepsilon)\} ds, \\ x_{k+1}(\tau, \varepsilon) &= X_{m-1}(\tau) I_1 (I - P_0) c_k^{(0)} + \varepsilon G_1(\tau) P_0 c_k^{(1)} + x_{k+1}^{(2)}(\tau, \varepsilon), \\ z_{k+1}(\tau, \varepsilon) &= z_0(\tau, \bar{c}^*) + x_{k+1}(\tau, \varepsilon), \quad k = 0, 1, 2, \ldots; \quad x_0(\tau, \varepsilon) = x_0^{(2)}(\tau, \varepsilon) = 0. \end{aligned} \tag{22}$$

Condition (21) for the existence of a T-periodic solution of system (4) is drawn up with the aid of the nonlinearity $Z(z_0 + x, \varepsilon)$ and the first approximation $x_1(\tau, \varepsilon)$ to the sought-for solution. It is a necessary and sufficient condition for existence of a T-periodic solution of the system that serves to define a second approximation to the sought-for solution.

5. Example. To illustrate the algorithm for studying autonomous systems presented above, we consider a problem concerning the existence of periodic solutions of the system

$$\dot{z} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} z + \varepsilon \begin{bmatrix} 0 \\ (1 - z_1^2) z_2 \end{bmatrix} = Az + \varepsilon Z(z), \tag{23}$$

where $z = \mathrm{col}\,(z_1, z_2) \in R^2$. System (23) describes the oscillations of a vacuum tube oscillator under "mild" conditions [8]. Using the notation above, we have

$$X_m(\tau) = \begin{bmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{bmatrix}, \quad H(\tau) = \begin{bmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{bmatrix}, \quad X_{m-1}(\tau) = \begin{bmatrix} \cos\tau \\ -\sin\tau \end{bmatrix}.$$

Equation (8) for generating amplitudes of system (23) has the form

$$F(c) = \bar{c} \int_0^{2\pi} H(s) \left\{ \alpha \begin{bmatrix} -\sin s \\ -\cos s \end{bmatrix} + \begin{bmatrix} 0 \\ -(1 - \bar{c}^2 \cos^2 s) \sin s \end{bmatrix} \right\} ds = -\pi\bar{c} \begin{bmatrix} \frac{1}{4}(\bar{c}^2 - 4) \\ 2\alpha \end{bmatrix} = 0. \tag{24}$$

We select as a root of Eq. (24) the vector $c^* = \mathrm{col}\,(\bar{c}^*, \alpha^*) \in R^2$ with components $\bar{c}^* = 2$, $\alpha^* = 0$. We then easily construct the matrices

$$\bar{A}_1(\tau) = \begin{bmatrix} 0 & -2\sin\tau \\ \sin\tau\,(12\cos^2\tau - 1) & 2\cos\tau \end{bmatrix},$$

$$B_0 = \int_0^{2\pi} H(s)\bar{A}_1(s)\,ds = -2\pi \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Since $\det B_0 \neq 0$, we have the critical case of the first order, i.e., $\bar{c}^* = 2$, $\alpha^* = 0$ are simple roots of Eq. (24) for generating amplitudes of the problem concerning periodic solutions of the autonomous system (23). Therefore, according to Theorem 2, system (23) has a unique periodic solution $T_1(\varepsilon) = 2\pi(1 + \varepsilon\alpha(\varepsilon))$, $\alpha(0) = \alpha^* = 0$, which reverts for $\varepsilon = 0$ to the 2π-periodic solution $z_0(\tau, \bar{c}^*)$.

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