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(MIN, MAX)-EQUIVALENCE OF POSETS AND NONNEGATIVE TITS FORMS

V. M. Bondarenko[1] **and M. V. Stepochkina**[1]

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We study the relationship between the (min, max)-equivalence of posets and properties of their quadratic Tits form related to nonnegative definiteness. In particular, we prove that the Tits form of a poset S is nonnegative definite if and only if the Tits form of any poset (min, max)-equivalent to S is weakly nonnegative.

1. Introduction

Let S be a finite poset that does not contain the element 0. The quadratic form $q_S \colon \mathbb{Z}^{S \cup 0} \to \mathbb{Z}$ of this poset defined by the equality

$$q_S(z) = z_0^2 + \sum_{i \in S} z_i^2 + \sum_{i<j,\, i,j \in S} z_i z_j - z_0 \sum_{i \in S} z_i$$

is called its Tits quadratic form. For the first time, this form was considered by Drozd [1], who showed that a poset S has a finite (representation) type over a field k if and only if its Tits form is weakly positive. It was shown in [2] that S has the tame type if and only if the Tits form is weakly nonnegative.

Positive Tits forms[2] and their applications in the theory of Tits representations were investigated in many works (see, e.g., [3–7]). The present paper is devoted to the study of posets with nonnegative Tits form.

We now recall the notion of the (min, max)-equivalence of posets [4].

For a minimal (respectively, maximal) element $a \in S,$ we denote by $S_a^{\uparrow}$ $\big($respectively, $S_a^{\downarrow}\big)$ the poset $T = T' \cup \{a\},$ where $T' = S \setminus \{a\}$ in the sense of posets (in this case, T and S are equal as ordinary sets) and the element a is already maximal (respectively, minimal); furthermore, a is comparable with x in T if and only if a is incomparable with x in $S.$ We write $S_{xy}^{\uparrow\uparrow}$ instead of $(S_x^{\uparrow})_y^{\uparrow},$ $S_{xy}^{\uparrow\downarrow}$ instead of $(S_x^{\uparrow})_y^{\downarrow},$ etc.

A poset T is called (min, max)-*equivalent* to a poset S if T is equal to a certain poset of the form

$$\overline{S} = S_{x_1 x_2 \ldots x_p}^{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_p}, \quad p \geq 0,$$

where $\varepsilon_i \in \{\uparrow, \downarrow\}$ and $x_i,$ $i \in \{1, \ldots, p\},$ is a minimal (respectively, maximal) element of $S_{x_1 x_2 \ldots x_{i-1}}^{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_{i-1}}$ if $\varepsilon_i = \uparrow$ $\big($respectively, $\varepsilon_i = \downarrow\big);$ for $p = 0,$ we assume that $\overline{S} = S.$ Moreover, the condition that the elements $x_1,$ $x_2, \ldots, x_p$ are different is not necessary.

In the case where all ε_i are equal to $\uparrow$ $\big($respectively, $\downarrow\big),$ we say that the poset T is min-*equivalent* (respectively, max-*equivalent*) to the poset $S.$ According to Corollary 2 and Proposition 11 in [6], the (min, max)-, min-, and max-equivalences are equivalence relations, and, furthermore, they are equivalent.

[1] Institute of Mathematics, Ukrainian National Academy of Sciences, Kiev, Ukraine.

[2] We use the term “positive form” instead of “positive-definite form” in connection with the conventional term “weakly positive form.” The same is also true for nonnegative forms.

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Note that one can naturally extend the notion of (min, max)-equivalence to the notion of (min, max)-isomorphism by assuming that the posets S and S' are (min, max)-*isomorphic* if there exists a poset T that is (min, max)-equivalent to S and isomorphic to S'; the same is true for the min-equivalence and max-equivalence.

We now formulate the main results of the present paper.

Recall that a quadratic form $f(z) = f(z_1, \ldots, z_m)\colon \mathbb{Z}^m \to \mathbb{Z}$ ($\mathbb{Z}$ is the set of all integers) is called *weakly nonnegative* if it takes a nonnegative value on any vector with nonnegative coordinates. A form that takes nonnegative values on all vectors is called *nonnegative* (see Remark 1); in this case, we write $f(z) \geq 0$.

A poset S is called *NP-critical* (respectively, *WNP-critical*) if the Tits form of any proper subset of it is nonnegative (respectively, weakly nonnegative), but the Tits form of S itself does not possess this property.

The aim of the present paper is to prove the following theorems:

Theorem 1. *For an arbitrary fixed poset S, the following assertions are true:*

(1) if the Tits form of any poset min-*equivalent to S is weakly nonnegative, then the Tits form of S itself is nonnegative;*

(2) if the Tits form of S is nonnegative, then the Tits form of any poset min-*equivalent to S is also nonnegative (and, a fortiori, it is weakly nonnegative).*

Theorem 2. *A poset S is NP-critical if and only if it is* min-*equivalent to a certain WNP-2pt-critical poset.*

In the conditions of Theorems 1 and 2, the min-equivalence can be replaced by the max-equivalence or by the (min, max)-equivalence (by virtue of their equivalence indicated above), as well as by the min-, max-, or (min, max)-isomorphism.

Note that *WNP*-critical posets (there are only six of them) are known (see Sec. 4). Theorem 2 gives an efficient method for the investigation of NP-critical sets.

Analogous results for positive and weakly positive Tits forms (along with many other results) were obtained by the authors in [6].

2. Definitions and Notation for Posets

Let $T = (T_0, \leq)$ be a poset. In what follows, a subset X of the poset T is always understood as a subset $X \subseteq T_0$ together with the induced relation of partial order, which will be denoted by the same symbol (in this case, for $x, y \in X,$ the notation "$x \leq y$ in T" is equivalent to the notation "$x \leq y$ in X"); one-element subsets are identified with elements themselves. For simplicity, we write $x \in T$ instead of $x \in T_0,$ $X \subset T$ instead of $X \subset T_0,$ etc. (these natural simplifications have been used in Introduction).

A subset X is called *lower* (respectively, *upper*) if $x \in X$ whenever $x < y$ (respectively, $x > y$) and $y \in X,$ and it is called *dense* if $x \in X$ whenever $y < x < z$ and $y, z \in X.$ It is obvious that lower and upper subsets are dense. Let $\overleftarrow{A}$ and $\overrightarrow{A},$ where A is a subset of $T,$ denote, respectively, the least lower subset and the least upper subset in T that contain $A.$ The subset $\overleftrightarrow{A} = \overleftarrow{A} \cap \overrightarrow{A},$ which is the least dense subset that contains $A,$ is called the *closure of the subset A in S.*

The notation $X < Y$ for subsets of T means that $x < y$ for any $x \in X$ and $y \in Y.$ Note that $Z < \varnothing$ and $\varnothing < Z$ for any subset $Z.$ Further, the notation $x \not\lessgtr y$ means that the elements x and y are incomparable. We set $T^{\not\lessgtr}(a) = \{x \in T \mid x \not\lessgtr a\}.$ For an element $a \in T,$ we denote by $\{a\}^{<}$ $\big($respectively, $\{a\}^{>}\big)$ the subset of all $x \in T$ for which $x < a$ (respectively, $x > a$).

The maximum number of pairwise incomparable elements of a poset T is called the *width* of this poset and is denoted by $w(T).$

We say that a poset T is the *sum* of subsets A and B and write $T = A + B$ if $T = A \cup B$ and $A \cap B = \varnothing$. If $A < B$, then this sum is called *ordinal*, and if $x \not\lessgtr y$ for any $x \in A$ and $y \in B$, then it is called *direct*. In the first case, we write $T = \{A < B\}$; in the second case, we write $T = A \coprod B$. These definitions can naturally be generalized to the case of an arbitrary number of subsets. A poset is called *primitive* if it is a direct sum of chains (linearly ordered sets).

3. Properties of min-Equivalent Posets

The min-equivalence of posets is denoted by $\cong_{\min}$ (the symbol $\cong$ denotes an isomorphism of posets). If $T_2 \cong_{\min} T_1$, then, by definition, T_2 and T_1 are equal as ordinary sets. Therefore, every subset $X \subset T_1$ is also a subset in T_2, but not necessarily with the same partial order. If the order relation on X has not been changed, then (to point out this fact) we often write X° instead of X $\big($for $X \subset T_2\big)$.

Let S be a poset. A finite sequence $\alpha = (x_1, x_2, \ldots, x_p)$ of elements $x_i \in S$ is called min-*admissible* if the expression $\overline{S} = S^{\uparrow\uparrow\ldots\uparrow}_{x_1 x_2 \ldots x_p}$ is meaningful (the case $p = 0$ is not excluded). In this case, we also write $\overline{S} = S^{\uparrow}_{\alpha}$.

Let $\mathcal{P}(S)$ denote the set of all min-admissible sequences and let $\mathcal{P}_1(S)$ denote the set of all sequences of this type without repetitions. Denote the subset of S that consists of all elements x_i of a sequence $\alpha \in \mathcal{P}_1(S)$ by $[\alpha]_S$. Note that if S and T are min-equivalent, then there does not always exist $\alpha \in \mathcal{P}_1(S)$ such that $T = S^{\uparrow}_{\alpha}$ (see Sec. 6 in [6]).

According to Corollary 5 in [6], $\mathcal{P}_1(S)$ contains a sequence α such that $[\alpha]_S = X$ if and only if the subset X is lower. According to Corollary 9 in [6], if $\alpha, \beta \in \mathcal{P}_1(S)$ and $[\alpha]_S = [\beta]_S$, then $S^{\uparrow}_{\alpha} = S^{\uparrow}_{\beta}$. Therefore, for the lower subset X, it is natural to define a poset $S^{\uparrow}_{X}$ by assuming that $S^{\uparrow}_{X} = S^{\uparrow}_{\alpha}$, where $\alpha \in \mathcal{P}_1(S)$ is an arbitrary sequence such that $[\alpha]_S = X$. It follows from Proposition 6 in [6] that, in $\overline{S} = S^{\uparrow}_{X}$, the subset X is already upper and, hence, $Y = S \setminus X$ is lower (with the same partial orders); moreover, $y < x$ for $y \in Y$ and $x \in X$ (in $\overline{S}$) if and only if $y \not\lessgtr x$ in S. In particular, if $S = X \coprod Y$ $\big($respectively, $S = \{X < Y\}\big)$, then $S^{\uparrow}_{X} = \{Y < X\}$ $\big($respectively, $S^{\uparrow}_{X} = X \coprod Y\big)$.

We now give several statements necessary for what follows. As above, S is an arbitrary poset. Let $M_-(S)$ $\big($respectively, $M_+(S)\big)$ denote the set of all its minimal (respectively, maximal) elements.

Lemma 1 (lemma on cyclic permutation)**.** *Let $X = R \coprod \{M < N\}$ be a subset of a poset S. Then there exist $T_1, T_2 \cong_{\min} S$ in which $X = M^\circ \coprod \{N^\circ < R^\circ\}$ and $X = N^\circ \coprod \{R^\circ < M^\circ\}$, respectively.*

Indeed, as T_1 and T_2, we can take the poset $T = S^{\uparrow}_{Y}$ for $Y = S \setminus \overrightarrow{N}$ and $Y = \overleftarrow{M}$, respectively.

Corollary 1. *If S contains subsets A and B such that $A < B$, then $A \cup B = A^\circ \coprod B^\circ$ in a certain $T \cong_{\min} S$.*

Indeed, one should set $M = A$, $N = B$, and $R = \varnothing$ in the conditions of the lemma.

Corollary 2. *Suppose that $L = L_1 \coprod \ldots \coprod L_m$ is a primitive subset of S $\big(L_1, \ldots, L_m$ are nonempty chains$\big)$ and c is an element of S such that $c > L_i$ for any $i \neq m$ and $\{c\}^< \cap L_m = \varnothing$. Then there exists $T_1 \cong_{\min} S$ that contains the primitive subset $L' = L_1^\circ \coprod \ldots \coprod L_{m-1}^\circ \coprod L'_m$, where L'_m is a chain of order $|L_m| + 1$ that contains L_m°.*

Indeed, the case $w(L) < 3$ is trivial. For $w(L) \geq 3$, one should use the lemma with $M = L_1 + \ldots + L_{m-1}$, $N = \{c\}$, and $R = L_m$.

Lemma 2. *Let L be a dense subset of S. Then there exists $T \cong_{\min} S$ in which L is a lower subset with the same partial order.*

Indeed, as $T,$ we can take $T = S_P^{\uparrow}$ for $P = \cup_{x \in M_-(L)} \{x\}^<.$

In conclusion of this section, we give one statement in the general case $\big($i.e., for sequences from $\mathcal{P}(S)\big);$ this statement was proved in [6] (Lemma 26).

Proposition 1. *Let* $\alpha = (x_1, x_2, \ldots, x_m) \in \mathcal{P}(S),$ *let* X *be a subset of* $S,$ *and let* α_X *be a subsequence of* α *that consists of all* $x_i \in X.$ *Then* $\alpha_X \in \mathcal{P}(X)$ *and* $X^{\uparrow}_{\alpha_X}$ *is a subset of* $S^{\uparrow}_{\alpha}.$

4. Properties of a Quadratic Tits Form Related to Its Nonnegativity

According to the main result of [4], quadratic Tits forms of min-equivalent posets are equivalent. In particular, this yields the following statement:

Proposition 2. *Let* S *and* T *be* min-*equivalent posets. Then their Tits forms are simultaneously either nonnegative or not.*

Recall that the ordinal sum $S = \{A_1 < A_2 < \ldots < A_s\}$ of antichains A_i of lengths 1 and 2 (an antichain of length m is a poset that consists of m pairwise incomparable elements) is called a *semichain*. This is equivalent to the statement that $w(S) < 3$ and S does not contain subsets of width 2 of the form $\{a\} \coprod \{b < c\}.$ The sets A_i are called the *links* of a semichain. If all links are one-element, then S is a chain.

Proposition 3. *If the poset* S *is a direct sum of two semichains, then its Tits form is nonnegative.*

Proof. By virtue of Proposition 2 and the lemma on cyclic permutation for $X = S$ and $M = \varnothing,$ it suffices to assume that S is a semichain; moreover, we can obviously assume that all its links are two-element. Thus, let $S = \{A_1 < A_2 < \ldots < A_s\},$ where $A_i = \{i^-, i^+\}.$ It is easy to see that

$$2q_S(z) = z_0^2 + \sum_{i=1}^{s} (z_{i^-} - z_{i^+})^2 + \left(z_0 - \sum_{j \in S} z_j \right)^2,$$

which implies that the form $q_S(z)$ is nonnegative.

Finally, we give a statement on the nonnegativity of Tits forms for several specific posets necessary in what follows.

Lemma 3. *The quadratic Tits form is nonnegative for the following posets:*

$$S_1 = \big\{1 \prec 5,\ 2 \prec 6,\ 3 \prec 7,\ 4 \prec 8,\ 1 \prec 6,\ 2 \prec 7,\ 3 \prec 8,\ 4 \prec 5\big\},$$

$$S_2 = \big\{2 \prec 5,\ 3 \prec 6,\ 4 \prec 7,\ 2 \prec 6,\ 3 \prec 7,\ 4 \prec 5\big\},$$

$$S_3 = \big\{2 \prec 5,\ 3 \prec 6,\ 4 \prec 7,\ 1 \prec 5,\ 1 \prec 6,\ 1 \prec 7\big\},$$

$$S_4 = \big\{2 \prec 4,\ 5 \prec 6 \prec 7 \prec 8 \prec 9,\ 3 \prec 4,\ 3 \prec 6\big\},$$

$$S_5 = \big\{2 \prec 5 \prec 6,\ 4 \prec 7 \prec 8,\ 3 \prec 5,\ 3 \prec 7\big\},$$

$$S_6 = \{1 \prec 4,\ 2 \prec 5,\ 6 \prec 7 \prec 8 \prec 9,\ 2 \prec 4,\ 3 \prec 5,\ 3 \prec 7\},$$

$$S_7 = \{1 \prec 3,\ 2 \prec 3,\ 4 \prec 6,\ 5 \prec 6,\ 2 \prec 7,\ 4 \prec 7,\ 7 \prec 8\},$$

$$S_8 = \{1 \prec 3 \prec 4,\ 6 \prec 7 \prec 8,\ 2 \prec 3,\ 2 \prec 9,\ 5 \prec 7,\ 5 \prec 9\},$$

$$S_9 = \{1 \prec 4 \prec 7,\ 2 \prec 5 \prec 8,\ 3 \prec 6 \prec 9,\ 1 \prec 8,\ 2 \prec 9,\ 3 \prec 7\},$$

$$S_{10} = \{1 \prec 2,\ 3 \prec 4,\ 5 \prec 6 \prec 7 \prec 8 \prec 9,\ 3 \prec 7\},$$

$$S_{11} = \{1 \prec 2,\ 3 \prec 4 \prec 5,\ 6 \prec 7 \prec 8 \prec 9,\ 1 \prec 5,\ 3 \prec 8\},$$

$$S_{12} = \{1 \prec 2,\ 3 \prec 4 \prec 5 \prec 6,\ 7 \prec 8 \prec 9,\ 1 \prec 5,\ 3 \prec 9\},$$

$$S_{13} = \{1 \prec 2 \prec 3,\ 4 \prec 5 \prec 6,\ 7 \prec 8 \prec 9,\ 5 \prec 8\},$$

$$S_{14} = \{2 \prec 3 \prec 4,\ 5 \prec 6 \prec 7 \prec 8 \prec 9,\ 2 \prec 8\}.$$

It is assumed in the conditions of the lemma that each of the sets S_i consists of the elements $1, 2, \ldots, s$, where s is the maximal number contained in its definition in explicit form.

The nonnegativity of the quadratic Tits form for the indicated posets was proved in [8] (see Lemma 4.3).

5. *WNP*-Critical Posets

Let $\langle p \rangle$ denote the chain $1 < 2 < \ldots < p$ and let $\langle p, q, \ldots, r \rangle$ denote the direct sum of the chains $\langle p \rangle$, $\langle q \rangle, \ldots, \langle r \rangle$. We set $N = \{1 \prec 2,\ 3 \prec 4,\ 1 \prec 4\}$.

Proposition 4. *A poset is WNP-critical if and only if it is isomorphic to one of the following posets*: $\mathcal{N}_1 = \langle 1,1,1,1,1 \rangle$, $\mathcal{N}_2 = \langle 1,1,1,2 \rangle$, $\mathcal{N}_3 = \langle 2,2,3 \rangle$, $\mathcal{N}_4 = \langle 1,3,4 \rangle$, $\mathcal{N}_5 = \langle 1,2,6 \rangle$, *and* $\mathcal{N}_6 = N \coprod \langle 5 \rangle$.

Proof. It follows from Theorem A in [2] and Proposition 3 in [1] that, first, any poset with not weakly nonnegative Tits form contains a certain $\mathcal{N}_i$ as a subset and, second, any proper subset of each $\mathcal{N}_i$ has a weakly nonnegative Tits form. In the proof of Theorem B in [2], it was shown that the Tits form of each $\mathcal{N}_i$ is not weakly nonnegative. These three facts imply that the proposition is true.

For the first time, the posets $\mathcal{N}_1 - \mathcal{N}_6$ were introduced in Nazarova's work [9] devoted to the description of tame posets, and, therefore, we call them *Nazarova critical sets*. Their subsets $\mathcal{K}_1 = \langle 1,1,1,1 \rangle$, $\mathcal{K}_2 = \langle 2,2,2 \rangle$, $\mathcal{K}_3 = \langle 1,3,3 \rangle$, $\mathcal{K}_4 = \langle 1,2,5 \rangle$, and $\mathcal{K}_5 = N \coprod \langle 4 \rangle$ are called *Kleiner critical sets*; they were introduced in [10] and play the same role as the Nazarova sets, but in the description of posets of finite type.

In the case where P is a given poset (say, $P = \mathcal{K}_i$ or $P = \mathcal{N}_i$), we say that a poset T contains P if T contains X isomorphic to P; if, in addition, $T = P$, then we say that T is of the form P.

The statements presented below follow directly from definitions.

Lemma 4. *The closure of a nondense subset of the form* $\mathcal{K}_i$ *contains a certain* $\mathcal{N}_j$.

Lemma 5. *If a primitive poset T contains a certain $\mathcal{K}_i$ as a proper subset, then it contains a certain $\mathcal{N}_j$.*

Using the last lemma and Corollaries 1 and 2, we obtain the following statement:

Lemma 6. *If a poset S contains a certain primitive $K = \mathcal{K}_i$ and $x \in S$ is an element such that $K' = K \cap \{x\}^<$ has the width $w \geq w(S) - 1$ and is selected as a direct summand from K (in particular, coincides with K), then there exists $T \cong_{\min} S$ that contains a certain $\mathcal{N}_j$.*

One can obtain this statement by using Corollary 1 with $A = K$ and $B = x$ if $w(K') = w(S)$ (taking into account that $K' = K$ in this case) and Corollary 2 with $L = K$ and $L_m = K \setminus K'$ if $w(K') = w(S) - 1$ and then applying Lemma 5.

We now prove the following statement:

Proposition 5. *Any WNP-critical poset is NP-critical.*

Proof. By definition, the Tits form of a *WNP*-critical set is not nonnegative. Further, using Proposition 4, one can easily show that any maximal subset M of every *WNP*-critical set is either a subset (not necessarily proper) of a certain Kleiner critical set or a direct sum of two semichains the total number of two-element links of which does not exceed 1. In the first case, the Tits form of the set M is nonnegative by virtue of Lemma 4.3 in [8]. In the second case, this statement is true by virtue of Proposition 3 (according to Proposition 21 in [6], the Tits form is positive in this case).

6. Theorem on Posets without *WNP*-Critical Subsets

Consider posets such that any posets min-equivalent to them do not contain Nazarova critical sets. Denote the collection of all these posets by $\mathcal{F}$.

The key role in the proof of Theorems 1 and 2 is played by the following statement:

Theorem 3. *The Tits form of a set $S \in \mathcal{F}$ is nonnegative.*

Note that it suffices to prove Theorem 3 for any fixed poset min-equivalent to S. We use this fact in what follows, choosing the most suitable poset in each individual case.

We now pass to the proof of Theorem 3. It is obvious that $w(S) \leq 4$ $\big($otherwise $S \supset \mathcal{N}_1\big)$. If any poset $T \cong_{\min} S$ does not contain Kleiner critical sets, then, according to Proposition 24 in [6], the Tits form of the poset S is positive. For this reason, we assume that S contains at least one $\mathcal{K} \cong \mathcal{K}_i$, $1 \leq i \leq 5$, and, furthermore, $S \neq \mathcal{K}$ because, by virtue of Lemma 3, the posets $\mathcal{K}_i$ have nonnegative Tits forms.

First, we consider the case where $\mathcal{K} \cong \mathcal{K}_1$.

By virtue of Lemma 2 for $L = \mathcal{K}_1$, we can assume that $\mathcal{K} = M_-(S)$. Let $\mathcal{K} = \{a_1, a_2, a_3, a_4\}$. Denote the subset $\{a_i\}^> \cap \{a_j\}^>$ by L_{ij}. In what follows, since $L_{ji} = L_{ij}$, considering these sets we always assume, for convenience, that $i < j$. Since $w(S) = 4$ and $S \not\supseteq \mathcal{N}_2$, the union of all $\widehat{L}_{ij} = L_{ij} \cup \{a_i, a_j\}$ is equal to S. Furthermore, by virtue of Lemma 6, the subsets L_{ij} and L_{pq} do not intersect for $(i,j) \neq (p,q)$. Then each L_{ij} is a semichain (possibly empty) because otherwise $\mathcal{K} \cup L_{ij}$ contains $\mathcal{N}_1$ or $\mathcal{N}_2$, depending on whether L_{ij} contains the subset $X \cong \langle 1,1,1 \rangle$ or $Y \cong \langle 1,2 \rangle$.

If only one of the semichains L_{ij} (of width 1 or 2) is nonempty or only two semichains L_{ij} and L_{pq} are nonempty for $\{i,j\} \cap \{p,q\} = \varnothing$, then S is a direct sum of two semichains, and, by virtue of Proposition 3, we have $q_S(z) \geq 0$. This is also true for the case where there exists at least one L_{ij} that is a semichain of width 2; indeed, in this case, each L_{pq} is empty for $|\{i,j\} \cap \{p,q\}| = 1$ because otherwise a subset that consists of two incomparable elements $a, b \in L_{ij}$, any element $c \in L_{pq}$, and elements of the subset $\mathcal{K} \setminus \{a_i, a_j\}$ (of order 2) is of the form $\mathcal{N}_2$.

Thus, for $\mathcal{K} \cong \mathcal{K}_1$, it remains to consider the case where each L_{ij} is a chain (possibly empty) and, furthermore, all L_{ij} are pairwise disjoint and there exist $L_{pq}, L_{rs} \neq \varnothing$ such that $|\{p,q\} \cap \{r,s\}| = 1$. We set $l_{ij} = |L_{ij}|$ and denote the number of nonempty L_{ij} by $m = m(S)$.

Assume that, in this case, one of the following conditions is satisfied:

(a) $m = 4$;

(b) $m = 3$ and, for (pairwise different and nonempty) L_{ij}, L_{pq}, and L_{rs}, one has

$$\big|\{i,j\} \cap \{p,q\}\big| = 1, \quad \big|\{p,q\} \cap \{r,s\}\big| = 1, \quad \big|\{i,j\} \cap \{r,s\}\big| = 1, \quad \big|\{i,j\} \cap \{p,q\} \cap \{r,s\}\big| = 0;$$

(c) $m = 3$ and, for (pairwise different and nonempty) L_{ij}, L_{pq}, and L_{rs}, one has

$$\big|\{i,j\} \cap \{p,q\} \cap \{r,s\}\big| = 1.$$

Then (up to renumbering of minimal elements) one of the following cases takes place:

(1.1) $l_{12} = l_{23} = l_{34} = l_{14} = 1$;

(1.2) $l_{12} \geq 1$, $l_{23} \geq 1$, $l_{34} \geq 1$, and $l_{14} > 1$;

(2.1) $l_{12} = l_{23} = l_{13} = 1$;

(2.2) $l_{12} \geq 1$, $l_{23} \geq 1$, and $l_{13} > 1$;

(3.1) $l_{12} = l_{13} = l_{14} = 1$;

(3.2) $l_{12} \geq 1$, $l_{13} \geq 1$, and $l_{14} > 1$.

Here, cases (1.1) and (1.2) correspond to condition (a), cases (2.1) and (2.2) correspond to condition (b), and cases (3.1) and (3.2) correspond to condition (c). Note that l_{ij} not mentioned here are assumed to be zero.[3]

If none of conditions (a)–(c) is satisfied, then, up to renumbering of minimal elements, one has either $m = 2$ and $L_{23}, L_{34} \neq \varnothing$ or $m = 3$ and $L_{12}, L_{23}, L_{34} \neq \varnothing$. In these cases, we set $l = (l_{23}, l_{34})$ and $l = (l_{12}, l_{23}, l_{34})$, respectively, and assume (for special posets) that several coordinates of the vector l can be defined not by a certain number but by inequalities of the form $> z$ and $\geq z$, where z is a certain natural number, and by more usual inequalities of the form $z_1 \leq s \leq z_2$. It is easy to see that, in this situation, one of the following cases takes place:

(4.1) $l = (1, 1 \leq s \leq 4)$;

(4.2) $l = (1, > 4)$;

(5.1) $l = (2, 2)$;

(5.2) $l = (\geq 2, > 2)$;

(6.1) $l = (1, 1, 1 \leq s \leq 3)$;

(6.2) $l = (1, 1, > 3)$;

[3] This assumption is also used in the investigation of the case $\mathcal{K} \cong \mathcal{K}_i$ for $i > 1$.

(7.1) $l = (1, 2, 1)$;

(7.2) $l = (1, > 2, 1)$;

(8.1) $l = (2, 1, 2)$;

(8.2) $l = (\geq 2, 1, > 2)$;

(9) $l = (\geq 1, > 1, > 1)$.

Let us analyze cases (1.1)–(9).

In cases $(i.1)$, $i = 1, 2, \ldots, 8$, the poset S is contained, up to an isomorphism, in S_i (see Lemma 3). In cases (1.2) and (2.2), S contains $\mathcal{N}_2$; in cases (3.2), (7.2), and (9), it contains $\mathcal{N}_3$; in cases (5.2) and (8.2), it contains $\mathcal{N}_4$; in case (4.2), it contains $\mathcal{N}_5$; and in case (6.2), it contains $\mathcal{N}_6$. With regard for Lemma 3, this implies that if $S \in \mathcal{F}$, then its Tits form is nonnegative.

Now let $\mathcal{K} \cong \mathcal{K}_i$ for $i > 1$. We assume that none of $T \cong_{\min} S$ contains $\mathcal{K}_1$ because the case $\mathcal{K} \cong \mathcal{K}_1$ has already been considered. Then, according to Corollary 1, the poset T does not contain subsets of the form $Q_{13} = \{R_1 < R_3\}$, $Q_{31} = \{R_3 < R_1\}$, and $Q_{22} = \{R_2 < R_2'\}$, where $R_1 \cong \langle 1 \rangle$ is a set that consists of a single element u_0, $R_2 \cong \langle 1, 1 \rangle$ $\big($respectively, $R_2' \cong \langle 1, 1 \rangle\big)$ is a set that consists of two incomparable elements u_1 and u_2 $\big($respectively, u_1' and $u_2'\big)$, and $R_3 \cong \langle 1, 1, 1 \rangle$ is a set that consists of three pairwise incomparable elements v_1, v_2, and v_3.

Further, according to Lemma 4, the subset $\mathcal{K}$ is dense. Then, by virtue of Lemma 2 for $L = \mathcal{K}_i$, we can assume that $\mathcal{K}$ is a lower subset of S. In particular, this yields $M_-(\mathcal{K}) = M_-(S)$. We set $M_-(\mathcal{K}) = \{a_1, a_2, a_3\}$ and $M_+(\mathcal{K}) = \{b_1, b_2, b_3\}$ and assume that $a_1 \leq b_1$, $a_2 < b_2$, and $a_3 < b_3$.

First, we consider the case $\mathcal{K} \cong \mathcal{K}_i$ for $i \neq 5$.

We set $B_{ij} = \{b_i\}^> \cap \{b_j\}^>$ and $L_{ij} = \{a_i\}^> \cap \{b_j\}^>$ (considering them only for $i \neq j$); we also set $C_i = \{b_i\}^< \cup b_i$. According to Lemma 5, $\mathcal{K}$ is a maximal primitive subset both in S itself and in every $T \cong_{\min} S$ in which $\mathcal{K} \cong \mathcal{K}_i$. Then, by virtue of Lemma 6, we have $B_{ij} = \varnothing$, and, hence, $S \setminus \mathcal{K}$ is the union of all subsets L_{ij} $\big($otherwise S contains $\mathcal{K}_1\big)$, which are pairwise disjoint $\big($otherwise $S \supset Q_{31}\big)$. Furthermore, if L_{ij} is nonempty, then L_{is} for $j \neq s$ and L_{ji} are empty (otherwise $S \supset Q_{13}$ and $S \supset Q_{22}$, respectively). It follows from the relations $B_{ij} = \varnothing$ and $w(S) = 3$ that L_{ij} is a chain.

As in the case $\mathcal{K} \cong \mathcal{K}_1$, we denote the number of nonempty L_{ij} by $m = m(S)$ and set $l_{ij} = |L_{ij}|$.

First, let $\mathcal{K} \cong \mathcal{K}_2$. If $m = 3$, then, up to rearrangement of the numbers 1, 2, and 3 in subscripts, we get one of the following cases:

(10.1) $l_{12} = l_{23} = l_{31} = 1$;

(10.2) $l_{12} \geq 1$, $l_{23} \geq 1$, and $l_{31} > 1$.

If $m = 1, 2$, then one of the following cases (in which all l_{ij} that are not mentioned are zero) takes place:

(11.1) $1 \leq l_{12} \leq 3$;

(11.2) $l_{12} > 3$;

(12.1) $l_{12} = 1$ and $l_{23} = 2$;

(12.2) $l_{12} \geq 1$ and $l_{23} > 2$;

(13.1) $l_{12} = 2$ and $l_{23} = 1$;

(13.2) $l_{12} > 2$ and $l_{23} \geq 1$.

Now let us analyze cases (10.1)–(13.2).

In cases $(i.1)$, $i = 10, \dots, 13$, the poset S is contained, up to an isomorphism, in S_{i-1} (see Lemma 3). In cases (10.2), (11.2), (12.2), and (13.2), S is contained in $\mathcal{N}_3$, $\mathcal{N}_5$, $\mathcal{N}_6$, and $\mathcal{N}_4$, respectively. With regard for Lemma 3, this implies that if $S \in \mathcal{F}$, then its Tits form is nonnegative.

Now let $\mathcal{K} \cong \mathcal{K}_3$. In this case, we can assume that $T \cong_{\min} S$ does not contain $\mathcal{K}_2$ because the case $\mathcal{K} \cong \mathcal{K}_2$ has already been considered. According to the notation introduced above, $M_-(\mathcal{K}) = \{a_1, a_2, a_3\}$ and $M_+(\mathcal{K}) = \{b_1, b_2, b_3\}$, where $a_1 = b_1$, $a_2 < b_2$, and $a_3 < b_3$. Let c_2 and c_3 denote the "missing" elements of the subset $\mathcal{K}$: $a_2 < c_2 < b_2$ and $a_3 < c_3 < b_3$.

Note that the set $K_{ij} = \{c_i\}^> \cap \{b_j\}^>$ is empty if $i \neq j$ and $i, j \neq 1$ because otherwise, according to Corollary 2, for $L = L_1 \coprod L_2 \coprod L_3$, $L_1 = \{a_i, c_i\}, L_2 = \{a_j, c_j, b_j\}$, and $L_3 = \{a_1\}$, a certain $T_1 \cong_{\min} S$ contains $\mathcal{N}_3$. Further, L_{i1}, $i = 2, 3$, coincides with K_{i1}, otherwise $\mathcal{K} \cup (L_{i1} \setminus K_{i1})$ contains $\mathcal{N}_3$. In this situation, if $L_{i1} \neq \varnothing$, then $m = 1$ because, in the case where $L_{ij} \neq \varnothing$, $j \neq 1$, the subset $\mathcal{K} \cup L_{i1} \cup L_{ij}$ contains Q_{13}, and in the case where $L_{ji} \neq \varnothing$, $j \neq 1$, it contains $\mathcal{K}_2$.

Therefore, up to rearrangement of the numbers 2 and 3 in subscripts, one of the following cases takes place:

(14.1) $l_{21} \leq 2$;

(14.2) $l_{21} > 2$;

(15.1) $l_{23} \leq 2$;

(15.2) $l_{23} > 2$.

In cases (14.1) and (15.1), the poset S, up to an isomorphism, is contained in S_{13} and S_{14}, respectively (see Lemma 3). In cases (14.2) and (15.2), S contains $\mathcal{N}_4$ and $\mathcal{N}_5$, respectively. Thus, for $S \in \mathcal{F}$, its Tits form is nonnegative.

We now show that, in the case $\mathcal{K} \cong \mathcal{K}_4$, there exists $T \cong_{\min} S$ that contains $\mathcal{K}_2$ or $\mathcal{K}_3$ (the corresponding cases have already been considered). According to the notation introduced above, we have $M_-(\mathcal{K}) = \{a_1, a_2, a_3\}$ and $M_+(\mathcal{K}) = \{b_1, b_2, b_3\}$, where $a_1 = b_1$, $a_2 < b_2$, and $a_3 < b_3$. Let c_3, d_3, and e_3 denote the "missing" elements of the subset $\mathcal{K}$: $a_3 < c_3 < d_3 < e_3 < b_3$.

The subset L_{23} is empty because otherwise, if f denotes the maximal element of L_{23}, then $S_P^{\uparrow}$ with $P = S \setminus f$ contains $\mathcal{N}_6$ $\big($more exactly, $\mathcal{K} \cup f$ is of the form $\mathcal{N}_6\big)$. If $L_{32} \neq \varnothing$ and $g \in L_{32}$, then $g > c_3$ because otherwise the subset $(\mathcal{K} \setminus a_3) \cup g$ is of the form $\mathcal{N}_4$; then, according to Corollary 2 for $L = L_1 \coprod L_2 \coprod L_3$, $L_1 = \{a_3, c_3\}$, $L_2 = C_2$, and $L_3 = a_1$, there exists $T_1 \cong_{\min} S$ in which $\mathcal{K} \cup g$ is of the form $\mathcal{K}_2$. If $L_{31} \neq \varnothing$ and $h \in L_{31}$, then $h > d_3$ because otherwise $(\mathcal{K} \setminus \{a_3, c_3\} \cup h$ is of the form $\mathcal{N}_3$. Then, according to Corollary 2 for $L = L_1 \coprod L_2 \coprod L_3$, $L_1 = C_1, L_2 = \{a_3, c_3, d_3\}$, and $L_3 = C_2$, there exists $T_1 \cong_{\min} S$ in which $\mathcal{K} \cup h$ is of the form $\mathcal{K}_3$. Finally, if $L_{21} \neq \varnothing$ and $t \in L_{21}$, then $\mathcal{K} \cup t$ is of the form $\mathcal{N}_6$.

It remains to consider the case where $\mathcal{K} \cong \mathcal{K}_5$.

Let U denote the subset of $\mathcal{K}$ that consists of the elements a_1, b_1, a_2, and b_2 and, furthermore, let $a_1 < b_2$. Denote the "missing" elements of $\mathcal{K}$ by c_3 and d_3, assuming that $c_3 < d_3$. Then $\mathcal{K} = U \coprod C_3$, where $C_3 = \{a_3 < c_3 < d_3 < b_3\}$. We set $C_1 = \{a_1, b_1\}$ and $C_2 = \{a_2, b_2\}$.

We need a statement that concretizes Corollary 2 (in the generality required for our purposes) and obviously follows from its proof.

Corollary 3. *Suppose that* S, $L = L_1 \coprod \dots \coprod L_m$, *and* c *are the same as in the conditions of Corollary 2,* $m = 3$, $|L_1| = i$, $|L_2| = j$, $|L_3| = \max(i, j) - 1$, $i \leq j$, *and* $i + j = 4$. *Then there exists* $T_1 \cong_{\min} S$ *that contains* $\mathcal{K}_j$.

Let us show that the case $\mathcal{K} \cong \mathcal{K}_5$ reduces to the considered cases $\mathcal{K} \cong \mathcal{K}_2$ and $\mathcal{K} \cong \mathcal{K}_3$, namely, that there exists $T \cong_{\min} S$ that contains $\mathcal{K}_2$ or $\mathcal{K}_3$.

We now assume that this is not true, i.e., that every poset T min-equivalent to S contains neither $\mathcal{K}_2$ nor $\mathcal{K}_3$, and show that this leads to a contradiction.

First, we show that S is decomposable (with respect to the direct sum defined above). Assume that this is not true. Then there exists x such that $\{x\}^< \cap U \neq \varnothing$ and $\{x\}^< \cap C_3 \neq \varnothing$. Therefore, $x > a_3$. We set $R = \{x\}^< \cap U$. It is obvious that $b_2 \notin R$ (otherwise $\mathcal{K} \cup x$ contains Q_{31}). For the same reason, R cannot contain the elements a_1 and a_2 (respectively, b_1 and a_2) simultaneously. Furthermore, if $a_1 \in R$, then $b_1 \in R$, otherwise $\mathcal{K} \cup x$ contains Q_{13}. Thus, there are only two possibilities for R: (a) $R = C_1$ and (b) $R = \{a_2\}$. Case (a) is impossible because, for $x \not\gtrless c_3$, the subset $\mathcal{K} \cup x$ contains $\mathcal{K}_3$, and for $x > c_3$, by virtue of Corollary 3 for $L_1 = C_1$, $L_2 = \{a_3, c_3\}$, $L_3 = \{a_2\}$, and $c = x$, there exists $T_1 \cong_{\min} S$ that contains $\mathcal{K}_2$. Case (b) is also impossible because, for $x \not\gtrless b_3$, the subset $M_+(\mathcal{K}) \cup x$ is of the form $\mathcal{K}_1$, and for $x > b_3$, by virtue of Corollary 3 for $L_1 = a_2$, $L_2 = C_3 \setminus b_3$, $L_3 = C_1$, and $c = x$, there exists $T_1 \cong_{\min} S$ that contains $\mathcal{K}_3$ $\big($it is easy to see that the proof of Corollary 2 implies that there even exists $T_1 \cong_{\min} S$ that contains $\mathcal{N}_4\big)$.

Thus, S is decomposable into a direct sum of two proper subsets. It is clear that one of them contains U and the other contains C_3. Therefore, there exists x such that either $\{x\}^< \cap U = \varnothing$ and $\{x\}^< \cap C_3 \neq \varnothing$ or, vice versa, $\{x\}^< \cap U \neq \varnothing$ and $\{x\}^< \cap C_3 = \varnothing$. In the first case, for $x \not\gtrless b_3$, the subset $M_+(\mathcal{K}) \cup x$ is of the form $\mathcal{K}_1$. For $x > b_3$, the subset $\mathcal{K} \cup x$ is of the form $\mathcal{N}_6$. Let us show that the second case is also impossible. We set $V = T^{\not\gtrless}(x) \cap U$. It is easy to see that V is a subset of U of width $w \leq 1$ $\big($otherwise $\mathcal{K} \cup x$ contains $\mathcal{K}_1\big)$; furthermore, V is an upper subset because the subset $U \setminus V = \{x\}^< \cap U$ is lower. In the case $w = 1$, the subset $\mathcal{K} \cup x$ also contains Q_{22} if $V = \{b_2\}$ and $\mathcal{N}_4$ if $V = \{b_1\}$ or $V = C_2$. If V is empty, then, according to the lemma on cyclic rearrangement $\big($for $M = U$, $N = x$, and $R = C_3\big)$, there exists $T_1 \cong_{\min} S$ in which $\mathcal{K} \cup x$ is of the form $\mathcal{N}_6$.

Thus, we arrive at a contradiction. Therefore, there exists $T \cong_{\min} S$ that contains $\mathcal{K}_2$ or $\mathcal{K}_3$.

Theorem 3 is proved.

7. Proof of Theorems 1 and 2

We can now easily prove Theorems 1 and 2.

First, we prove Theorem 2. If the poset S is min-equivalent to the *WNP*-critical set $\mathcal{N}$, then, by virtue of Propositions 2 and 5, the Tits form $q_S(z)$ is not nonnegative. It is easy to see that Proposition 1, with regard for Propositions 2 and 5, implies that every proper subset $R \subset S$ has a nonnegative Tits form. Indeed, otherwise $\mathcal{N}$ has a proper subset $Q \cong_{\min} R$ whose Tits form is not nonnegative, which contradicts the fact that the set $\mathcal{N}$ is *NP*-critical. Thus, S is *NP*-critical.

Conversely, if S is *NP*-critical, then, according to Theorem 3, it is min-equivalent to a certain poset S' that contains a *WNP*-critical set $N \cong \mathcal{N}_i$. In this case, again by virtue of Propositions 1 and 2, we have $S' = N$, and, hence, S is min-equivalent to N.

We now pass to the proof of Theorem 1. Assertion (2) of the theorem follows directly from Proposition 2. If S satisfies the condition of assertion (1), then any poset min-equivalent to S does not contain *WNP*-critical subsets (by virtue of the definition of the latter). Therefore, according to Theorem 3, S has a nonnegative Tits form.

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