

DESCRIPTION OF POSETS CRITICAL WITH RESPECT TO THE NONNEGATIVITY OF THE QUADRATIC TITS FORM

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UDC 512.64+512.56

We present the complete description of finite posets whose Tits form is not nonnegative but all proper subsets of which have nonnegative Tits forms. A similar result for positive forms was obtained by the authors earlier.

1. Introduction

In [1], for a finite quiver (oriented graph) Q with the set of vertices Q_0 and the set of arrows Q_1 , Gabriel introduced a quadratic form $q_Q: \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z}$ and called it the quadratic Tits form of the quiver Q :

$$q_Q(z) = \sum_{i \in Q_0} z_i^2 - \sum_{i \rightarrow j} z_i z_j,$$

where $i \rightarrow j$ runs over the set Q_1 . In [1], he also proved that the quiver has a finite representation type over the field k (i.e., finitely many indecomposable representations, to within an isomorphism) if and only if its Tits form is positive. This Gabriel's work laid the foundations of a new field of algebra dealing with the investigation of the relationships between the properties of representations of various objects and the properties of quadratic forms associated with these objects.

Later, the quadratic Tits forms for quivers with relations and posets were studied by Brenner [2] and Drozd [3], respectively. In the general case of matrix problems without relations, the Tits form was introduced by Kleiner and Roiter [4].

In [3], it is shown that a poset has a finite representation type if and only if its Tits form is weakly positive, i.e., takes positive value on any nonzero vector with nonnegative coordinates (representations of posets were introduced by Nazarova and Roiter in [5]). Note that the Tits form of a quiver is weakly positive if and only if it is positive. For posets, this is not true. Therefore, the investigations of posets with positive Tits form seems to be quite natural (at least, from the formal point of view). Posets of this sort were studied in [6–14]. Moreover, in the cited works, the posets with positive Tits form were also studied from the viewpoint of their properties directly connected with the categories of representations.

The present paper is devoted to the investigation of posets with nonnegative Tits form and continues our investigation of posets of this sort originated in [15].

2. Formulation of the Main Result

Let S be a finite poset that does not contain element 0. Its quadratic Tits form is defined as a quadratic form $q_S: \mathbb{Z}^{S \cup 0} \rightarrow \mathbb{Z}$ given by the equality

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Translated from *Ukrains'kyi Matematychnyi Zhurnal*, Vol. 61, No. 5, pp. 611–624, May, 2009. Original article submitted December 23, 2008.

$$q_S(z) = z_0^2 + \sum_{i \in S} z_i^2 + \sum_{i < j, i, j \in S} z_i z_j - z_0 \sum_{i \in S} z_i.$$

A poset S is called critical with respect to positivity (nonnegativity) of the quadratic Tits form or, briefly, *P-critical* (*NP-critical*) if the Tits form of any its proper subset is positive (negative) but the Tits form of the entire S is not a form of the same type.

The *width* of a poset S is defined as the maximum number of its pairwise incomparable elements. A poset dual to S is denoted by S^{op} . In other words, $S^{\text{op}} = S$ as ordinary sets and, moreover $x < y$ in S^{op} if and only if $x > y$ in S . The posets S and T are called *antiisomorphic* if S is isomorphic to T^{op} .

The list of all *P-critical* posets can be found in [10]. This list is also presented in Table 1 (see the last section). The aim of the present paper is to describe all *NP-critical* sets .

The following theorem is true:

Theorem 1. *The NP-critical posets are exhausted, to within isomorphisms and antiisomorphisms, by posets 1–115 presented in Table 2 (see Section 5).*

We now make several comments on Table 2.

The poset with number i in Table 2 is denoted by N_i . If the set N_i has width 2 and, in Table 2, it is indicated that $i = j'$, then this means that N_i can be obtained from N_j by replacing its unique maximum point with the unique minimum point. The same is true for the case $i = j'' = (j')'$ (it is necessary to compare N_i with $N_{j'}$). If the set N_i has width 3 and, in Table 2, it is indicated that $i = j'$, then this means that the procedure described above should be applied not to N_i and N_j but to their connected components (direct terms) of width 2. The same is true for the case $i = j''$.

Any posets S and T obtained from each other as a result of operations of this sort are called *0-isomorphic*. Excluding the sets with the numbers $i = j'$ and $i = j''$ from Table 2, we obtain the description of *NP-critical* sets to within 0-isomorphism and duality.

The same is true for Table 1.

3. Min-Equivalent Posets

We now present several facts about min-equivalent posets. This type of equivalence was introduced by one of the authors (V.B.) in [7]. Its detailed investigation was carried out in [10].

Let S be a poset. For the minimum element $a \in S$, by S_a^\uparrow we denote a poset that coincides with S as a set with the same relation of ordering on $S \setminus \{a\}$ but, in S_a^\uparrow , the element a is already the maximum element and, in addition, a is comparable with x in T if and only if a is incomparable with x in S . We write $S_{xy}^{\uparrow\uparrow}$ instead of $(S_x^\uparrow)^\uparrow_y$, $S_{xyz}^{\uparrow\uparrow\uparrow}$ instead of $((S_x^\uparrow)^\uparrow_y)^\uparrow_z$, etc.

A poset T is called *min-equivalent* to a poset S if T is equal to a certain poset of the form

$$\bar{S} = S_{x_1 x_2 \dots x_p}^{\uparrow\uparrow \dots \uparrow}, \quad p \geq 0,$$

where x_i is the minimum element of $S_{x_1 x_2 \dots x_{i-1}}^{\uparrow\uparrow \dots \uparrow}$ for any $i \in \{1, \dots, p\}$. For $p = 0$, we assume that $\bar{S} = S$. It is not required that the elements x_1, x_2, \dots, x_p must be different.

The min-equivalence of posets is denoted by \cong_{min} .

Note that the notion of min-equivalence can be naturally extended to the notion of min-isomorphism if we assume that the posets S and S' are *min-isomorphic* if there exists a poset T min-equivalent to S and isomorphic to S' .

Let S be a poset. A finite sequence $\alpha = (x_1, x_2, \dots, x_p)$ of elements $x_i \in S$ is called *min-admissible* if the expression $\bar{S} = S_{x_1 x_2 \dots x_p}^{\uparrow \dots \uparrow}$ is meaningful (the case $p = 0$ is not excluded). In this case, we also write $\bar{S} = S_\alpha^\uparrow$.

The set of all min-admissible sequences is denoted by $\mathcal{P}(S)$ and the set of all sequences of this sort without repetitions by $\mathcal{P}_1(S)$. Let $[\alpha]_S$ be a subset in S formed by all elements x_i of the sequence $\alpha \in \mathcal{P}_1(S)$. Note that, for the min-equivalent posets S and T , a sequence $\alpha \in \mathcal{P}_1(S)$ such that $T = S_\alpha^\uparrow$ does not always exist (see [10], Sec. 6).

A subset X is called *lower* if $x \in X$ whenever $x < y$ and $y \in X$. The notation $X < Y$ for the subsets S means that $x < y$ for any $x \in X$ and $y \in Y$. Note that $Z < \emptyset$ and $\emptyset < Z$ for any subset Z .

According to Corollaries 5 and 9 from [10], the following two assertions are true:

- (1) in $\mathcal{P}_1(S)$, there exists a sequence α such that $[\alpha]_S = X$ if and only if X is a lower subset;
- (2) if $\alpha, \beta \in \mathcal{P}_1(S)$ and $[\alpha]_S = [\beta]_S$, then $S_\alpha^\uparrow = S_\beta^\uparrow$.

Hence, for the lower subset X , one can naturally define the poset S_X^\uparrow by setting $S_X^\uparrow = S_\alpha^\uparrow$, where $\alpha \in \mathcal{P}_1(S)$ is an arbitrary sequence such that $[\alpha]_S = X$. It is easy to see (Proposition 6 in [10]) that $a < b$ in $\bar{S} = S_X^\uparrow$ if and only if one of the following conditions is satisfied:

- (a) $a < b$ in S and either $a, b \in X$ or $a, b \notin X$;
- (b) $a \asymp b$ in S and $b \in X, a \notin X$.

This, in particular, implies that if Z is a lower subset in X such that $Z < S \setminus X$, then Z is a lower subset also in S_X^\uparrow .

An algorithm for the description of all posets min-equivalent to a fixed poset S is proposed (and justified) in [10]. This algorithm consists of the following steps:

- I. To describe all lower subsets $X \neq S$ in S and construct the poset S_X^\uparrow for each of these sets.
- II. To describe all couples (Y, X) formed by a proper lower subset Y in S and a nonempty lower subset X in Y such that $X < S \setminus Y$ and construct, for each couple of this type, the poset $S_{YX}^{\uparrow \uparrow} = (S_Y^\uparrow)_X^\uparrow$.
- III. To select a single poset from each class of isomorphic sets in the families of posets constructed in Steps I and II.

Subsets X and X' indicated in Step I are called *strongly isomorphic* if there exists an automorphism $\varphi: S \rightarrow S$ such that $\varphi(X) = X'$ (as subposets). Similarly, two couples (Y, X) and (Y', X') indicated in Step II are called *strongly isomorphic* if there exists an automorphism $\varphi: S \rightarrow S$ such that $\varphi(Y) = Y'$ and $\varphi(X) = X'$. Clearly, it suffices to describe the subsets in Step I and the couples of subsets in Step II to within a strong isomorphism.

4. Proof of Theorem 1

First, we formulate one of the two main results obtained in [15] and used in what follows.

It is known that the following posets (in [15], they are called the critical Nazarova posets) play an important role in the theory of representations:

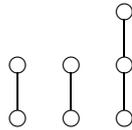
$\mathcal{N}_1 = \{1, 2, 3, 4, 5\}$, where all elements are pairwise incomparable:



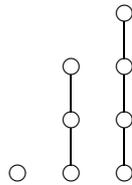
$$\mathcal{N}_2 = \{1, 2, 3, 4, 5 \mid 4 \prec 5\} :$$



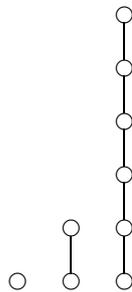
$$\mathcal{N}_3 = \{1, 2, 3, 4, 5, 6, 7 \mid 1 \prec 2, 3 \prec 4, 5 \prec 6 \prec 7\} :$$



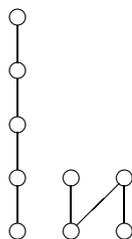
$$\mathcal{N}_4 = \{1, 2, 3, 4, 5, 6, 7, 8 \mid 2 \prec 3 \prec 4, 5 \prec 6 \prec 7 \prec 8\} :$$



$$\mathcal{N}_5 = \{1, 2, 3, 4, 5, 6, 7, 8, 9 \mid 2 \prec 3, 4 \prec 5 \prec 6 \prec 7 \prec 8 \prec 9\} :$$



$$\mathcal{N}_6 = \{1, 2, 3, 4, 5, 6, 7, 8, 9 \mid 1 \prec 2 \prec 3 \prec 4 \prec 5, 6 \prec 7, 8 \prec 9, 6 \prec 9\} :$$



We can now formulate the required result from [15].

Theorem 2. *A poset S is NP-critical if and only if it is min-equivalent to a critical Nazarova poset.*

Thus, all NP -critical posets can be obtained as follows: It is necessary to take all critical Nazarova sets and describe all posets min-equivalent to each of these sets (in what follows, we always say “all” instead of “all to within an isomorphism”). However, if we use solely the definition of min-equivalence, then it is not completely clear how to realize the corresponding process in practice (because, from the formal point of view, it is infinite). To do this, we use the algorithm outlined in the previous section.

We now directly proceed to the proof of Theorem 1.

Step I. We describe (to within a strong isomorphism) all lower subsets in the critical Nazarova sets $\mathcal{N}_1 - \mathcal{N}_6$ (see Sec. 4):

for \mathcal{N}_1 $A_{1,1} = \emptyset$, $A_{1,2} = \{1\}$, $A_{1,3} = \{1, 2\}$, $A_{1,4} = \{1, 2, 3\}$, and $A_{1,5} = \{1, 2, 3, 4\}$;

for \mathcal{N}_2 $A_{2,1} = \emptyset$, $A_{2,2} = \{1\}$, $A_{2,3} = \{4\}$, $A_{2,4} = \{1, 2\}$, $A_{2,5} = \{1, 4\}$, $A_{2,6} = \{4, 5\}$, $A_{2,7} = \{1, 2, 3\}$, $A_{2,8} = \{1, 2, 4\}$, $A_{2,9} = \{1, 4, 5\}$, $A_{2,10} = \{1, 2, 3, 4\}$, and $A_{2,11} = \{1, 2, 4, 5\}$;

for \mathcal{N}_3 $A_{3,1} = \emptyset$, $A_{3,2} = \{1\}$, $A_{3,3} = \{5\}$, $A_{3,4} = \{1, 2\}$, $A_{3,5} = \{1, 3\}$, $A_{3,6} = \{1, 5\}$, $A_{3,7} = \{5, 6\}$, $A_{3,8} = \{1, 2, 3\}$, $A_{3,9} = \{1, 2, 5\}$, $A_{3,10} = \{1, 3, 5\}$, $A_{3,11} = \{1, 5, 6\}$, $A_{3,12} = \{5, 6, 7\}$, $A_{3,13} = \{1, 2, 3, 4\}$, $A_{3,14} = \{1, 2, 3, 5\}$, $A_{3,15} = \{1, 2, 5, 6\}$, $A_{3,16} = \{1, 3, 5, 6\}$, $A_{3,17} = \{1, 5, 6, 7\}$, $A_{3,18} = \{1, 2, 3, 4, 5\}$, $A_{3,19} = \{1, 2, 3, 5, 6\}$, $A_{3,20} = \{1, 2, 5, 6, 7\}$, $A_{3,21} = \{1, 3, 5, 6, 7\}$, $A_{3,22} = \{1, 2, 3, 4, 5, 6\}$, and $A_{3,23} = \{1, 2, 3, 5, 6, 7\}$;

for \mathcal{N}_4 $A_{4,1} = \emptyset$, $A_{4,2} = \{1\}$, $A_{4,3} = \{2\}$, $A_{4,4} = \{5\}$, $A_{4,5} = \{1, 2\}$, $A_{4,6} = \{1, 5\}$, $A_{4,7} = \{2, 3\}$, $A_{4,8} = \{2, 5\}$, $A_{4,9} = \{5, 6\}$, $A_{4,10} = \{1, 2, 3\}$, $A_{4,11} = \{1, 2, 5\}$, $A_{4,12} = \{1, 5, 6\}$, $A_{4,13} = \{2, 3, 4\}$, $A_{4,14} = \{2, 3, 5\}$, $A_{4,15} = \{2, 5, 6\}$, $A_{4,16} = \{5, 6, 7\}$, $A_{4,17} = \{1, 2, 3, 4\}$, $A_{4,18} = \{1, 2, 3, 5\}$, $A_{4,19} = \{1, 2, 5, 6\}$, $A_{4,20} = \{1, 5, 6, 7\}$, $A_{4,21} = \{2, 3, 4, 5\}$, $A_{4,22} = \{2, 3, 5, 6\}$, $A_{4,23} = \{2, 5, 6, 7\}$, $A_{4,24} = \{5, 6, 7, 8\}$, $A_{4,25} = \{1, 2, 3, 4, 5\}$, $A_{4,26} = \{1, 2, 3, 5, 6\}$, $A_{4,27} = \{1, 2, 5, 6, 7\}$, $A_{4,28} = \{1, 5, 6, 7, 8\}$, $A_{4,29} = \{2, 3, 4, 5, 6\}$, $A_{4,30} = \{2, 3, 5, 6, 7\}$, $A_{4,31} = \{2, 5, 6, 7, 8\}$, $A_{4,32} = \{1, 2, 3, 4, 5, 6\}$, $A_{4,33} = \{1, 2, 3, 5, 6, 7\}$, $A_{4,34} = \{1, 2, 5, 6, 7, 8\}$, $A_{4,35} = \{2, 3, 4, 5, 6, 7\}$, $A_{4,36} = \{2, 3, 5, 6, 7, 8\}$, $A_{4,37} = \{1, 2, 3, 4, 5, 6, 7\}$, $A_{4,38} = \{1, 2, 3, 5, 6, 7, 8\}$, and $A_{4,39} = \{2, 3, 4, 5, 6, 7, 8\}$;

for \mathcal{N}_5 $A_{5,1} = \emptyset$, $A_{5,2} = \{1\}$, $A_{5,3} = \{2\}$, $A_{5,4} = \{4\}$, $A_{5,5} = \{1, 2\}$, $A_{5,6} = \{1, 4\}$, $A_{5,7} = \{2, 3\}$, $A_{5,8} = \{2, 4\}$, $A_{5,9} = \{4, 5\}$, $A_{5,10} = \{1, 2, 3\}$, $A_{5,11} = \{1, 2, 4\}$, $A_{5,12} = \{1, 4, 5\}$, $A_{5,13} = \{2, 3, 4\}$, $A_{5,14} = \{2, 4, 5\}$, $A_{5,15} = \{4, 5, 6\}$, $A_{5,16} = \{1, 2, 3, 4\}$, $A_{5,17} = \{1, 2, 4, 5\}$, $A_{5,18} = \{1, 4, 5, 6\}$, $A_{5,19} = \{2, 3, 4, 5\}$, $A_{5,20} = \{2, 4, 5, 6\}$, $A_{5,21} = \{4, 5, 6, 7\}$, $A_{5,22} = \{1, 2, 3, 4, 5\}$, $A_{5,23} = \{1, 2, 4, 5, 6\}$, $A_{5,24} = \{1, 4, 5, 6, 7\}$, $A_{5,25} = \{2, 3, 4, 5, 6\}$, $A_{5,26} = \{2, 4, 5, 6, 7\}$, $A_{5,27} = \{4, 5, 6, 7, 8\}$, $A_{5,28} = \{1, 2, 3, 4, 5, 6\}$, $A_{5,29} = \{1, 2, 4, 5, 6, 7\}$, $A_{5,30} = \{1, 4, 5, 6, 7, 8\}$, $A_{5,31} = \{2, 3, 4, 5, 6, 7\}$, $A_{5,32} = \{2, 4, 5, 6, 7, 8\}$, $A_{5,33} = \{4, 5, 6, 7, 8, 9\}$, $A_{5,34} = \{1, 2, 3, 4, 5, 6, 7\}$, $A_{5,35} = \{1, 2, 4, 5, 6, 7, 8\}$, $A_{5,36} = \{1, 4, 5, 6, 7, 8, 9\}$, $A_{5,37} = \{2, 3, 4, 5, 6, 7, 8\}$, $A_{5,38} = \{2, 4, 5, 6, 7, 8, 9\}$, $A_{5,39} = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A_{5,40} = \{1, 2, 4, 5, 6, 7, 8, 9\}$, and $A_{5,41} = \{2, 3, 4, 5, 6, 7, 8, 9\}$;

for \mathcal{N}_6 $A_{6,1} = \emptyset$, $A_{6,2} = \{1\}$, $A_{6,3} = \{6\}$, $A_{6,4} = \{8\}$, $A_{6,5} = \{1, 2\}$, $A_{6,6} = \{1, 6\}$, $A_{6,7} = \{1, 8\}$, $A_{6,8} = \{6, 7\}$, $A_{6,9} = \{6, 8\}$, $A_{6,10} = \{1, 2, 3\}$, $A_{6,11} = \{1, 2, 6\}$, $A_{6,12} = \{1, 2, 8\}$, $A_{6,13} = \{1, 6, 7\}$, $A_{6,14} = \{1, 6, 8\}$, $A_{6,15} = \{6, 7, 8\}$, $A_{6,16} = \{6, 8, 9\}$, $A_{6,17} = \{1, 2, 3, 4\}$, $A_{6,18} = \{1, 2, 3, 6\}$, $A_{6,19} = \{1, 2, 3, 8\}$, $A_{6,20} = \{1, 2, 6, 7\}$, $A_{6,21} = \{1, 2, 6, 8\}$, $A_{6,22} = \{1, 6, 7, 8\}$, $A_{6,23} = \{1, 6, 8, 9\}$, $A_{6,24} = \{6, 7, 8, 9\}$, $A_{6,25} = \{1, 2, 3, 4, 5\}$, $A_{6,26} = \{1, 2, 3, 4, 6\}$, $A_{6,27} = \{1, 2, 3, 4, 8\}$, $A_{6,28} = \{1, 2, 3, 6, 7\}$, $A_{6,29} = \{1, 2, 3, 6, 8\}$, $A_{6,30} = \{1, 2, 6, 7, 8\}$, $A_{6,31} = \{1, 2, 6, 8, 9\}$, $A_{6,32} = \{1, 6, 7, 8, 9\}$, $A_{6,33} = \{1, 2, 3, 4, 5, 6\}$, $A_{6,34} =$

$\{1, 2, 3, 4, 5, 8\}$, $A_{6,35} = \{1, 2, 3, 4, 6, 7\}$, $A_{6,36} = \{1, 2, 3, 4, 6, 8\}$, $A_{6,37} = \{1, 2, 3, 6, 7, 8\}$, $A_{6,38} = \{1, 2, 3, 6, 8, 9\}$, $A_{6,39} = \{1, 2, 6, 7, 8, 9\}$, $A_{6,40} = \{1, 2, 3, 4, 5, 6, 7\}$, $A_{6,41} = \{1, 2, 3, 4, 5, 6, 8\}$, $A_{6,42} = \{1, 2, 3, 4, 6, 7, 8\}$, $A_{6,43} = \{1, 2, 3, 4, 6, 8, 9\}$, $A_{6,44} = \{1, 2, 3, 6, 7, 8, 9\}$, $A_{6,45} = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A_{6,46} = \{1, 2, 3, 4, 5, 6, 8, 9\}$, and $A_{6,47} = \{1, 2, 3, 4, 6, 7, 8, 9\}$.

By $N_{i,j}$ we denote the poset S_X^\uparrow for $S = \mathcal{N}_i$ and $X = A_{i,j}$. Then it is easy to see that $\mathcal{N}_{1,1} \cong N_{115}$, $\mathcal{N}_{1,2} \cong N_{114}$, $\mathcal{N}_{1,3} \cong N_{45}$, $\mathcal{N}_{1,4} \cong N_{45}^{\text{op}}$, $\mathcal{N}_{1,5} \cong N_{114}^{\text{op}}$, $\mathcal{N}_{2,1} \cong N_{112}$, $\mathcal{N}_{2,2} \cong N_{43}$, $\mathcal{N}_{2,3} \cong N_{113}$, $\mathcal{N}_{2,4} \cong N_1^{\text{op}}$, $\mathcal{N}_{2,5} \cong N_{44}$, $\mathcal{N}_{2,6} \cong N_{41}$, $\mathcal{N}_{2,7} \cong N_{41}^{\text{op}}$, $\mathcal{N}_{2,8} \cong N_{44}^{\text{op}}$, $\mathcal{N}_{2,9} \cong N_1$, $\mathcal{N}_{2,10} \cong N_{113}^{\text{op}}$, $\mathcal{N}_{2,11} \cong N_{43}^{\text{op}}$, $\mathcal{N}_{3,1} \cong N_{46}$, $\mathcal{N}_{3,2} \cong N_{49}$, $\mathcal{N}_{3,3} \cong N_{50}$, $\mathcal{N}_{3,4} \cong N_4$, $\mathcal{N}_{3,5} \cong N_{53}$, $\mathcal{N}_{3,6} \cong N_{52}$, $\mathcal{N}_{3,7} \cong N_{47}$, $\mathcal{N}_{3,8} \cong N_6^{\text{op}}$, $\mathcal{N}_{3,9} \cong N_7$, $\mathcal{N}_{3,10} \cong N_{54}^{\text{op}}$, $\mathcal{N}_{3,11} \cong N_{51}$, $\mathcal{N}_{3,12} \cong N_2$, $\mathcal{N}_{3,13} \cong N_2^{\text{op}}$, $\mathcal{N}_{3,14} \cong N_{51}^{\text{op}}$, $\mathcal{N}_{3,15} \cong N_7^{\text{op}}$, $\mathcal{N}_{3,16} \cong N_{54}$, $\mathcal{N}_{3,17} \cong N_6$, $\mathcal{N}_{3,18} \cong N_{47}^{\text{op}}$, $\mathcal{N}_{3,19} \cong N_{52}^{\text{op}}$, $\mathcal{N}_{3,20} \cong N_4^{\text{op}}$, $\mathcal{N}_{3,21} \cong N_{53}^{\text{op}}$, $\mathcal{N}_{3,22} \cong N_{50}^{\text{op}}$, $\mathcal{N}_{3,23} \cong N_{49}^{\text{op}}$, $\mathcal{N}_{4,1} \cong N_{55}$, $\mathcal{N}_{4,2} \cong N_{17}$, $\mathcal{N}_{4,3} \cong N_{62}$, $\mathcal{N}_{4,4} \cong N_{63}$, $\mathcal{N}_{4,5} \cong N_{15}^{\text{op}}$, $\mathcal{N}_{4,6} \cong N_{18}^{\text{op}}$, $\mathcal{N}_{4,7} \cong N_{58}$, $\mathcal{N}_{4,8} \cong N_{69}$, $\mathcal{N}_{4,9} \cong N_{60}$, $\mathcal{N}_{4,10} \cong N_{13}^{\text{op}}$, $\mathcal{N}_{4,11} \cong N_{66}^{\text{op}}$, $\mathcal{N}_{4,12} \cong N_{16}^{\text{op}}$, $\mathcal{N}_{4,13} \cong N_{11}$, $\mathcal{N}_{4,14} \cong N_{65}$, $\mathcal{N}_{4,15} \cong N_{68}$, $\mathcal{N}_{4,16} \cong N_{56}$, $\mathcal{N}_{4,17} \cong N_8^{\text{op}}$, $\mathcal{N}_{4,18} \cong N_{64}^{\text{op}}$, $\mathcal{N}_{4,19} \cong N_{67}^{\text{op}}$, $\mathcal{N}_{4,20} \cong N_{14}^{\text{op}}$, $\mathcal{N}_{4,21} \cong N_{14}$, $\mathcal{N}_{4,22} \cong N_{67}$, $\mathcal{N}_{4,23} \cong N_{64}$, $\mathcal{N}_{4,24} \cong N_8$, $\mathcal{N}_{4,25} \cong N_{56}^{\text{op}}$, $\mathcal{N}_{4,26} \cong N_{68}^{\text{op}}$, $\mathcal{N}_{4,27} \cong N_{65}^{\text{op}}$, $\mathcal{N}_{4,28} \cong N_{11}^{\text{op}}$, $\mathcal{N}_{4,29} \cong N_{16}$, $\mathcal{N}_{4,30} \cong N_{66}$, $\mathcal{N}_{4,31} \cong N_{13}$, $\mathcal{N}_{4,32} \cong N_{60}^{\text{op}}$, $\mathcal{N}_{4,33} \cong N_{69}^{\text{op}}$, $\mathcal{N}_{4,34} \cong N_{58}^{\text{op}}$, $\mathcal{N}_{4,35} \cong N_{18}$, $\mathcal{N}_{4,36} \cong N_{15}$, $\mathcal{N}_{4,37} \cong N_{63}^{\text{op}}$, $\mathcal{N}_{4,38} \cong N_{62}^{\text{op}}$, $\mathcal{N}_{4,39} \cong N_{17}^{\text{op}}$, $\mathcal{N}_{5,1} \cong N_{70}$, $\mathcal{N}_{5,2} \cong N_{27}$, $\mathcal{N}_{5,3} \cong N_{74}$, $\mathcal{N}_{5,4} \cong N_{84}^{\text{op}}$, $\mathcal{N}_{5,5} \cong N_{25}^{\text{op}}$, $\mathcal{N}_{5,6} \cong N_{29}$, $\mathcal{N}_{5,7} \cong N_{23}$, $\mathcal{N}_{5,8} \cong N_{95}$, $\mathcal{N}_{5,9} \cong N_{82}$, $\mathcal{N}_{5,10} \cong N_{19}^{\text{op}}$, $\mathcal{N}_{5,11} \cong N_{94}$, $\mathcal{N}_{5,12} \cong N_{31}$, $\mathcal{N}_{5,13} \cong N_{26}$, $\mathcal{N}_{5,14} \cong N_{96}$, $\mathcal{N}_{5,15} \cong N_{80}$, $\mathcal{N}_{5,16} \cong N_{71}^{\text{op}}$, $\mathcal{N}_{5,17} \cong N_{101}^{\text{op}}$, $\mathcal{N}_{5,18} \cong N_{30}^{\text{op}}$, $\mathcal{N}_{5,19} \cong N_{23}$, $\mathcal{N}_{5,20} \cong N_{99}$, $\mathcal{N}_{5,21} \cong N_{77}$, $\mathcal{N}_{5,22} \cong N_{77}^{\text{op}}$, $\mathcal{N}_{5,23} \cong N_{99}^{\text{op}}$, $\mathcal{N}_{5,24} \cong N_{28}^{\text{op}}$, $\mathcal{N}_{5,25} \cong N_{30}$, $\mathcal{N}_{5,26} \cong N_{101}$, $\mathcal{N}_{5,27} \cong N_{71}$, $\mathcal{N}_{5,28} \cong N_{80}^{\text{op}}$, $\mathcal{N}_{5,29} \cong N_{96}^{\text{op}}$, $\mathcal{N}_{5,30} \cong N_{26}^{\text{op}}$, $\mathcal{N}_{5,31} \cong N_{31}^{\text{op}}$, $\mathcal{N}_{5,32} \cong N_{94}^{\text{op}}$, $\mathcal{N}_{5,33} \cong N_{19}$, $\mathcal{N}_{5,34} \cong N_{82}^{\text{op}}$, $\mathcal{N}_{5,35} \cong N_{95}^{\text{op}}$, $\mathcal{N}_{5,36} \cong N_{23}^{\text{op}}$, $\mathcal{N}_{5,37} \cong N_{29}^{\text{op}}$, $\mathcal{N}_{5,38} \cong N_{25}$, $\mathcal{N}_{5,39} \cong N_{84}^{\text{op}}$, $\mathcal{N}_{5,40} \cong N_{74}^{\text{op}}$, $\mathcal{N}_{5,41} \cong N_{27}^{\text{op}}$, $\mathcal{N}_{6,1} \cong N_{85}$, $\mathcal{N}_{6,2} \cong N_{93}$, $\mathcal{N}_{6,3} \cong N_{76}$, $\mathcal{N}_{6,4} \cong N_{97}^{\text{op}}$, $\mathcal{N}_{6,5} \cong N_{91}$, $\mathcal{N}_{6,6} \cong N_{98}$, $\mathcal{N}_{6,7} \cong N_{111}$, $\mathcal{N}_{6,8} \cong N_{37}$, $\mathcal{N}_{6,9} \cong N_{75}^{\text{op}}$, $\mathcal{N}_{6,10} \cong N_{89}$, $\mathcal{N}_{6,11} \cong N_{100}$, $\mathcal{N}_{6,12} \cong N_{110}$, $\mathcal{N}_{6,13} \cong N_{39}$, $\mathcal{N}_{6,14} \cong N_{106}^{\text{op}}$, $\mathcal{N}_{6,15} \cong N_{104}^{\text{op}}$, $\mathcal{N}_{6,16} \cong N_{36}^{\text{op}}$, $\mathcal{N}_{6,17} \cong N_{86}$, $\mathcal{N}_{6,18} \cong N_{102}$, $\mathcal{N}_{6,19} \cong N_{109}$, $\mathcal{N}_{6,20} \cong N_{40}$, $\mathcal{N}_{6,21} \cong N_{107}^{\text{op}}$, $\mathcal{N}_{6,22} \cong N_{103}^{\text{op}}$, $\mathcal{N}_{6,23} \cong N_{108}^{\text{op}}$, $\mathcal{N}_{6,24} \cong N_{32}^{\text{op}}$, $\mathcal{N}_{6,25} \cong N_{32}$, $\mathcal{N}_{6,26} \cong N_{103}$, $\mathcal{N}_{6,27} \cong N_{108}$, $\mathcal{N}_{6,28} \cong N_{40}^{\text{op}}$, $\mathcal{N}_{6,29} \cong N_{107}$, $\mathcal{N}_{6,30} \cong N_{102}^{\text{op}}$, $\mathcal{N}_{6,31} \cong N_{109}$, $\mathcal{N}_{6,32} \cong N_{86}^{\text{op}}$, $\mathcal{N}_{6,33} \cong N_{104}$, $\mathcal{N}_{6,34} \cong N_{36}$, $\mathcal{N}_{6,35} \cong N_{39}^{\text{op}}$, $\mathcal{N}_{6,36} \cong N_{106}$, $\mathcal{N}_{6,37} \cong N_{100}^{\text{op}}$, $\mathcal{N}_{6,38} \cong N_{110}^{\text{op}}$, $\mathcal{N}_{6,39} \cong N_{89}^{\text{op}}$, $\mathcal{N}_{6,40} \cong N_{37}^{\text{op}}$, $\mathcal{N}_{6,41} \cong N_{75}$, $\mathcal{N}_{6,42} \cong N_{98}^{\text{op}}$, $\mathcal{N}_{6,43} \cong N_{111}^{\text{op}}$, $\mathcal{N}_{6,44} \cong N_{91}^{\text{op}}$, $\mathcal{N}_{6,45} \cong N_{76}^{\text{op}}$, $\mathcal{N}_{6,46} \cong N_{97}$, and $\mathcal{N}_{6,47} \cong N_{93}^{\text{op}}$.

Step II. We now describe (to within a strong isomorphism) all couples (Y, X) of proper lower subsets in the critical Nazarova sets \mathcal{N}_2 – \mathcal{N}_6 such that $X \subseteq Y$ and $X < S \setminus Y$ (for \mathcal{N}_1 , these couples are absent):

for \mathcal{N}_2 $B_{2,1} = (A_{2,10}, \{4\})$;

for \mathcal{N}_3 $B_{3,1} = (A_{3,18}, \{5\})$, $B_{3,2} = (A_{3,22}, \{5\})$, $B_{3,3} = (A_{3,22}, \{5, 6\})$, and $B_{3,4} = (A_{3,23}, \{3\})$;

for \mathcal{N}_4 $B_{4,1} = (A_{4,25}, \{5\})$, $B_{4,2} = (A_{4,32}, \{5\})$, $B_{4,3} = (A_{4,32}, \{5, 6\})$, $B_{4,4} = (A_{4,34}, \{2\})$, $B_{4,5} = (A_{4,37}, \{5\})$, $B_{4,6} = (A_{4,37}, \{5, 6\})$, $B_{4,7} = (A_{4,37}, \{5, 6, 7\})$, $B_{4,8} = (A_{4,38}, \{2\})$, and $B_{4,9} = (A_{4,38}, \{2, 3\})$;

for \mathcal{N}_5 $B_{5,1} = (A_{5,16}, \{4\})$, $B_{5,2} = (A_{5,22}, \{4\})$, $B_{5,3} = (A_{5,22}, \{4, 5\})$, $B_{5,4} = (A_{5,28}, \{4\})$, $B_{5,5} = (A_{5,28}, \{4, 5\})$, $B_{5,6} = (A_{5,28}, \{4, 5, 6\})$, $B_{5,7} = (A_{5,34}, \{4\})$, $B_{5,8} = (A_{5,34}, \{4, 5\})$, $B_{5,9} = (A_{5,34}, \{4, 5, 6\})$, $B_{5,10} = (A_{5,34}, \{4, 5, 6, 7\})$, $B_{5,11} = (A_{5,39}, \{4\})$, $B_{5,12} = (A_{5,39}, \{4, 5\})$, $B_{5,13} = (A_{5,39}, \{4, 5, 6\})$, $B_{5,14} = (A_{5,39}, \{4, 5, 6, 7\})$, $B_{5,15} = (A_{5,39}, \{4, 5, 6, 7, 8\})$, and $B_{5,16} = (A_{5,40}, \{2\})$;

for \mathcal{N}_6 $B_{6,1} = (A_{6,32}, \{1\})$, $B_{6,2} = (A_{6,39}, \{1\})$, $B_{6,3} = (A_{6,39}, \{1, 2\})$, $B_{6,4} = (A_{6,41}, \{6\})$, $B_{6,5} = (A_{6,44}, \{1\})$, $B_{6,6} = (A_{6,44}, \{1, 2\})$, $B_{6,7} = (A_{6,44}, \{1, 2, 3\})$, $B_{6,8} = (A_{6,45}, \{6\})$, $B_{6,9} =$

$(A_{6,45}, \{8\})$, $B_{6,10} = (A_{6,45}, \{6, 8\})$, $B_{6,11} = (A_{6,46}, \{6\})$, $B_{6,12} = (A_{6,47}, \{1\})$, $B_{6,13} = (A_{6,47}, \{1, 2\})$, $B_{6,14} = (A_{6,47}, \{1, 2, 3\})$, and $B_{6,15} = (A_{6,47}, \{1, 2, 3, 4\})$.

By $N'_{i,j}$ we denote the poset $(S_Y^\uparrow)_X^\uparrow$ for $S = N_i$ and $(Y, X) = B_{i,j}$. Then it is easy to see that $N'_{2,1} \cong N_{42}$, $N'_{3,1} \cong N_3^{\text{op}}$, $N'_{3,2} \cong N_{48}$, $N'_{3,3} \cong N_3$, $N'_{3,4} \cong N_5$, $N'_{4,1} \cong N_9^{\text{op}}$, $N'_{4,2} \cong N_{57}^{\text{op}}$, $N'_{4,3} \cong N_{10}$, $N'_{4,4} \cong N_{12}^{\text{op}}$, $N'_{4,5} \cong N_{61}$, $N'_{4,6} \cong N_{57}$, $N'_{4,7} \cong N_9$, $N'_{4,8} \cong N_{59}$, $N'_{4,9} \cong N_{12}$, $N'_{5,1} \cong N_{20}^{\text{op}}$, $N'_{5,2} \cong N_{72}^{\text{op}}$, $N'_{5,3} \cong N_{21}^{\text{op}}$, $N'_{5,4} \cong N_{78}^{\text{op}}$, $N'_{5,5} \cong N_{73}^{\text{op}}$, $N'_{5,6} \cong N_{22}$, $N'_{5,7} \cong N_{81}^{\text{op}}$, $N'_{5,8} \cong N_{79}$, $N'_{5,9} \cong N_{73}$, $N'_{5,10} \cong N_{21}$, $N'_{5,11} \cong N_{83}$, $N'_{5,12} \cong N_{81}$, $N'_{5,13} \cong N_{78}$, $N'_{5,14} \cong N_{72}$, $N'_{5,15} \cong N_{20}$, $N'_{5,16} \cong N_{24}$, $N'_{6,1} \cong N_{33}^{\text{op}}$, $N'_{6,2} \cong N_{87}^{\text{op}}$, $N'_{6,3} \cong N_{34}^{\text{op}}$, $N'_{6,4} \cong N_{35}$, $N'_{6,5} \cong N_{90}^{\text{op}}$, $N'_{6,6} \cong N_{88}$, $N'_{6,7} \cong N_{34}$, $N'_{6,8} \cong N_{105}$, $N'_{6,9} \cong N_{38}^{\text{op}}$, $N'_{6,10} \cong N_{35}^{\text{op}}$, $N'_{6,11} \cong N_{38}$, $N'_{6,12} \cong N_{92}$, $N'_{6,13} \cong N_{90}$, $N'_{6,14} \cong N_{87}$, and $N'_{6,15} \cong N_{33}$.

Step III. It is easy to see that each poset N_i and N_i^{op} , where $i = 1, 2, \dots, 115$, appears only once in Steps I and II (moreover, if $N_i^{\text{op}} \cong N_i$, then N_i is present but N_i^{op} is absent).

Theorem 1 is thus proved.

5. Tables

Table 1. P -Critical Posets

1	2	3	4=3'	5	6=5'
7	8	9	10	11=10'	12=10''
13	14=13'	15	16	17	18

Table 1
(continued)

19 	20 	21 	22=21' 	23=21'' 	24
25=24' 	26 	27=26' 	28 	29 	30
31 	32 	33 	34 	35 	36
37=36' 	38 	39 	40 	41 	42
43 	44=43' 	45=43'' 	46 	47 	48
49 	50=49' 	51 	52=51' 	53 	54
55 	56 	57 	58 	59 	60

Table 1
(continued)

61 	62 	63 	64 	65 	66
67 	68=67' 	69 	70=69' 	71 	72
73 		74 		75 	

Table 2. NP-Critical Posets

1 	2 	3=2' 	4 	5=4' 	6
7 	8 	9=8' 	10=8'' 	11 	12=11'

Table 2
(continued)

13 	14 	15 	16 	17 	18
19 	20=19' 	21=19'' 	22=19''' 	23 	24=23''
25 	26 	27 	28 	29 	30
31 	32 	33=32' 	34=32'' 	35 	36=35'
37 	38=37' 	39 	40 	41 	42=41'

Table 2
(continued)

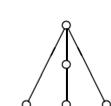
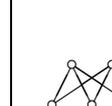
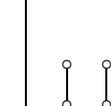
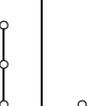
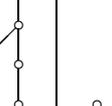
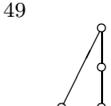
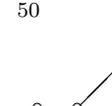
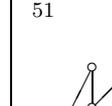
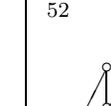
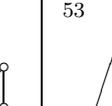
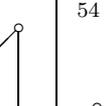
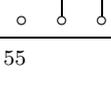
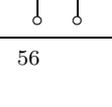
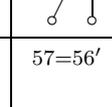
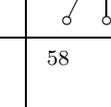
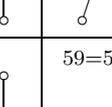
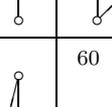
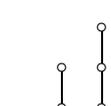
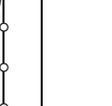
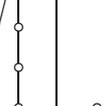
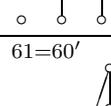
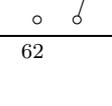
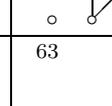
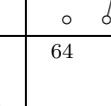
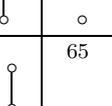
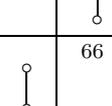
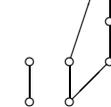
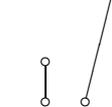
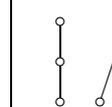
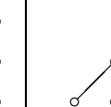
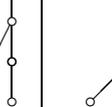
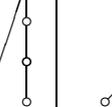
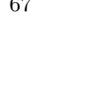
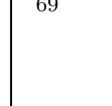
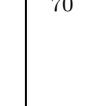
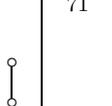
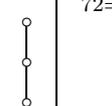
43 	44 	45 	46 	47 	48=47' 
49 	50 	51 	52 	53 	54 
55 	56 	57=56' 	58 	59=58' 	60 
61=60' 	62 	63 	64 	65 	66 
67 	68 	69 	70 	71 	72=71' 
73=72'' 	74 	75 	76 	77 	78=77' 
79=77'' 	80 	81=80' 	82 	83=82' 	84 

Table 2
(continued)

85 	86 	87=86' 	88=86'' 	89 	90=89'
91 	92=91' 	93 	94 	95 	96
97 	98 	99 	100 	101 	102
103 	104 	105 	106 	107 	108
109 	110 	111 	112 	113 	114
115 					

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